Math 501, Fall Term 2009
Final Exam

December 15^{th},2009

Do not turn this page over until instructed

- On the top of each exam booklet write
  1. Your student ID;
  2. The serial number of this exam;
  3. The number of the booklet (if you use more than one).
- On the sign-in sheet write
  1. Your student ID;
  2. The serial number of this exam;
  3. Your name.

Instructions

- You will have 150 minutes for this exam.
- No books, notes or electronic devices.
- Solutions should be written clearly, in complete English sentences, showing all your work.
- If you use a result from the lectures or the problem sets, quote it properly.
- Good luck!

1. (20 pts) Define the following terms
   (a) Sylow p-subgroup.
   (b) Solvable group.
   (c) Separable extension.
   (d) Algebraic closure.

2. (15 pts) Let G be a finite group of order $561 = 3 \cdot 11 \cdot 17$.
   (a) Show that G has unique subgroups A, B of order $11, 17$ respectively.
   (b) Show that A and B are central in G.
   (c) Show that G is cyclic.

3. (15 pts) Consider the polynomial $t^3 + 2t + 1$.
   (a) Show that it is irreducible in $\mathbb{F}_3[t]$.
   (b) Show that it is irreducible in $\mathbb{Q}[t]$.
   (c) Show that it is irreducible in $\mathbb{F}_{3^{2009}}[t]$.

4. (30 pts) Let p, q be primes and let $f(x) = x^p - q \in \mathbb{Z}[x]$.
   (a) Find a splitting field Σ for f over $\mathbb{Q}$.
   (b) Find $[\Sigma : \mathbb{Q}]$.
   (c) Find $\mathrm{Gal}(\Sigma : \mathbb{Q})$.

5. (10 pts) Let $\mathbb{Q} \subset K \subset L$ be number fields. Given $\alpha \in \mathcal{O}_L$ let $g \in K[x]$ be the minimal polynomial of α over K. Show that $g \in \mathcal{O}_K[x]$.

6. (10 pts) Let $r \in \mathbb{Q}$ and let $K = \mathbb{Q}(\cos(2\pi r))$.
   (a) Embed K in a radical extension of $\mathbb{Q}$.
   (b) Show that K is normal over $\mathbb{Q}$.
   (c) Show that $\mathrm{Gal}(K : \mathbb{Q})$ is Abelian.

   Hint: Express $\cos(2\pi r)$ using two roots of unity.