*This final exam has **8 questions** on **10 pages**, for a total of 102 marks.*

Duration: 2 hours 30 minutes

Full Name (including all middle names): ______________________________

Student-No: ______________________________

Signature: ______________________________

UBC Rules governing examinations:

1. Each candidate should be prepared to produce his/her library/AMS card upon request.
2. No candidate shall be permitted to enter the examination room after the expiration of one half hour, or to leave during the first half hour or the *last 15 minutes* of the examination. Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in the examination questions.
3. Candidates guilty of any of the following or similar practices shall be immediately dismissed from the examination, and shall be liable to disciplinary action:

   a) Making use of any books, papers or memoranda, other than those authorised by the examiners.

   b) Speaking or communicating with other candidates.

   c) Purposely exposing written papers to the view of other candidates. The plea of accident or forgetfulness will not be received.

Question:	1	2	3	4	5	6	7	8	Total
Points:	12	16	10	8	18	14	12	12	102
Score:									

Please read the following points carefully before starting to write.

- Continue on the back of the previous page if you run out of space.
- This is a closed-book examination. **None of the following are allowed**: documents, cheat sheets or electronic devices of any kind (including calculators, cell phones, etc.)
- **You may use any theorems discussed in class**

12 marks 1. Short answers:

(a) State carefully: Zorn's Lemma.

(b) Define carefully: A function $F : \mathbb{R} \to \mathbb{R}$ of bounded variation.

16 marks 2. Give examples of the following. Justify your examples carefully, using of course any theorems from the course.

(a) A sequence of Riemann integrable functions $\{f_n\}$ on $[0, 1]$ which converge pointwise to a bounded function f which is not Riemann integrable.

(b) A sequence of integrable functions $f_n : [0, 1] \to \mathbb{R}$ which converges in m-measure to an integrable function f, but not in L^1 to f. (Here all functions are $\mathcal{B}([0, 1])$-measurable and m is Lebesgue measure on $([0, 1], \mathcal{B}([0, 1]))$.)

$\boxed{\text{10 marks}}$ 3. Let $\{f_n\}$ be a sequence of measurable $\mathbb{R}$-valued functions on a measure space $(X, \mathcal{M}, \mu)$. We say f_n converges to a measurable function f almost uniformly iff for all $\varepsilon > 0$ there is an $E \in \mathcal{M}$ so that $\mu(E^c) < \varepsilon$ and

$$\lim_{n\to\infty} \sup_{x\in E} |f_n(x) - f(x)| = 0.$$

Prove that if f_n converges to f almost uniformly, then f_n converges to f μ-a.e.

8 marks 4. True or False. If true, give a proof. If false, provide, and justify, a counter-example.

If ν is a signed measure on $(X, \mathcal{M})$, then there is a set B in $\mathcal{M}$, such that

$$\nu(B) = \sup\{\nu(A) : A \in \mathcal{M}\}.$$

$\boxed{\text{18 marks}}$ 5. (a) Carefully state the Monotone Class Theorem (including the definition of a Monotone Class).

(b) Let μ_1 and μ_2 be finite measures on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ such that

$$\mu_1((a,b]) \leq \mu_2((a,b])$$

for all real numbers $a < b$.

i. Prove that $\mu_1(A) \leq \mu_2(A)$ for all Borel subsets A of $\mathbb{R}$.

ii. Prove that for any Borel measurable function $f : \mathbb{R} \to [0, \infty]$, $\int f d\mu_1 \leq \int f d\mu_2$.

14 marks

6. Let $f_1, f_2, \ldots$ be measurable $\mathbb{R}$-valued functions on the measure space $(X, \mathcal{M}, \mu)$, such that $\sum_{n=1}^{\infty} \int |f_n| d\mu < \infty$.

   (a) Prove that for μ-a.e. x, the series $\sum_{n=1}^{\infty} f_n(x)$ converges μ-a.e. to a function which is in $L^1(\mu)$.

   (b) Prove that $\int \sum_{n=1}^{\infty} f_n d\mu = \sum_{n=1}^{\infty} \int f_n d\mu$

12 marks 7. (a) Define carefully: an outer measure on a non-empty set X.

(b) Assume $\rho : \mathcal{P}(X) \to [0, \infty]$ is an outer measure on X and for $A \subset X$, define

$$\rho^*(A) = \inf\Big\{\sum_{j=1}^{\infty} \rho(E_j) : A \subset \cup_{j=1}^{\infty} E_j,\ E_j \subset X\Big\}.$$

Prove that $\rho^* = \rho$.

$\boxed{\text{12 marks}}$ 8. For $j = 1, 2$, let μ_j and ν_j be σ-finite measures on $(X_j, \mathcal{M}_j)$ such that $\nu_j \ll \mu_j$. Prove that $\nu_1 \times \nu_2 \ll \mu_1 \times \mu_2$ (as measures on $(X_1 \times X_2, \mathcal{M}_1 \otimes \mathcal{M}_2)$), and

$$\frac{d(\nu_1 \times \nu_2)}{d(\mu_1 \times \mu_2)}(x_1, x_2) = \frac{d\nu_1}{d\mu_1}(x_1)\frac{d\nu_2}{d\mu_2}(x_2) \qquad \mu_1 \times \mu_2 - a.e.$$

SPACE FOR MORE WORK