This final exam has ***4 questions*** *on* ***10 pages****, for a total of 100 marks.*

Duration: 2 hours 30 minutes

Full Name (Last, First, All middle names): ______________________________

Student-No: ______________________________ Course : 420 / 507 (circle one)

Signature: ______________________________

Question:	1	2	3	4	Total
Points:	48	16	16	20	100
Score:					

Please read the following points carefully before starting to write.

- Read all the questions carefully before starting to work.
- You should give complete arguments and explanations for all your answers and calculations; answers without justifications will not be marked.
- Continue on the back of the previous page if you run out of space.
- Attempt to answer all questions for partial credit.
- This is a closed-book examination. **None of the following are allowed**: documents, cheat sheets or electronic devices of any kind (including calculators, cell phones, etc.)
- You may not leave during the first 30 minutes or final 15 minutes of the exam.

48 marks

1. Short questions: 6 marks for each.

   (a) If X is a nonempty set and $\mu^* : \mathcal{P}(X) \to [0, \infty]$ is an outer measure, state the definition of a measurable set with respect to μ^*.

   (b) Let $(X, \mathcal{M}, \mu)$ be a measure space. If $X_1, X_2, \dots \in \mathcal{M}$ and $X_1 \supseteq X_2 \supseteq X_3 \supseteq \dots$, does it follow that $\mu(\bigcap_{j=1}^{\infty} X_j) = \lim_{j\to\infty} \mu(X_j)$? If yes, explain why; if no, give a counterexample.

(c) Let $(X, \mathcal{M}, \mu)$ be a measure space, and let f and f_n, $n = 1, 2, \ldots$, be measurable functions $X \to [0, \infty)$ such that $f_n(x) \searrow f(x)$ for all $x \in X$. Does it follow that $\int f_n d\mu \to \int f\, d\mu$? If yes, explain why; if no, give a counterexample.

(d) Let $(X, \mathcal{M}, \mu)$ be a measure space, and let f and f_n, $n = 1, 2, \ldots$, be functions in $L^1(\mu)$ such that $\int |f_n - f| d\mu \to 0$. Does it follow that $f_n(x) \to f(x)$ for almost all x? If yes, explain why; if no, give a counterexample.

(e) State Fatou's Lemma.

(f) Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be measure spaces, and let λ and ρ be signed measures on $\mathcal{M}$ and $\mathcal{N}$, respectively. If $\lambda \perp \mu$ and $\rho \perp \nu$, does it follow that $\lambda \times \rho \perp \mu \times \nu$? If yes, explain why; if no, give a counterexample.

(g) Give an example of a measure that is not σ-finite.

(h) Give an example of an algebra of sets that is not a σ-slgebra.

16 marks 2. Is there a non-zero Borel measure μ on $\mathbb{R}$, absolutely continuous with respect to the Lebesgue measure, such that $\mu(U)$ is integer for every open set U? If yes, give an example. If no, give a proof.

16 marks

3. Let $(X, \mathcal{M}, \mu)$ be a measure space, and let $f : X \to [0, \infty)$ be integrable with respect to μ. Let $d\nu = f\,d\mu$ (so that $f = \frac{d\nu}{d\mu}$). Prove that for any measurable function $g : X \to [0, \infty)$, we have $\int g\,d\nu = \int fg\,d\mu$. (Hint: start with simple functions.)

20 marks

4. (a) Let $f : \mathbb{R} \to [0, \infty)$ be measurable, and assume that $\int f\, dm < \infty$, where m is the Lebesgue measure. Prove that there is a nonincreasing function $g : [0, \infty) \to [0, \infty)$ such that

$$m(\{x : f(x) < a\}) = m(\{x : g(x) < a\}) \quad \text{for all} \quad a > 0. \tag{1}$$

(b) Prove that if $f, g : [0, 1] \to [0, \infty)$ are two measurable functions such that (1) holds, then $\int_{[0,1]} f\, dm = \int_{[0,1]} g\, dm$.

This page has been left blank for your workings and solutions.