Marks

[10] **1.** Define

(a) the completion of a measure

(b) measurable function

(c) mutually singular measures

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[15] **2.** Give an example of each of the following, together with a brief explanation of your example. If an example does not exist, explain why not.

(a) A measure which is semi–finite, but not σ–finite.

(b) A function $f : \mathbb{R} \to \mathbb{R}$ such that f is not Lebesgue measurable but $|f|$ is Lebesgue measurable.

(c) Two measures $\mu_1,\ \mu_2$ on the measurable space $(X, \mathcal{M})$ for which $\mu_1 - \mu_2$ is not a signed measure.

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[15] **3.** Let X be a metric space and $\mathcal{B}_X$ be the σ–algebra of Borel subsets of X. Let μ be a measure on $(X, \mathcal{B}_X)$ and μ^* be the outer measure determined by μ. Prove that if V_1 and V_2 are two disjoint Borel subsets of X and $E_1 \subset V_1$ and $E_2 \subset V_2$, then

$$\mu^*(E_1 \cup E_2) = \mu^*(E_1) + \mu^*(E_2)$$

Note that E_1 and E_2 need not be Borel.

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[15] **4.** Let $(X, \mathcal{M}, \mu)$ be a measure space and $E \in \mathcal{M}$. Prove, directly from the definitions of the two integrals, that

$$\int_E f \; d\mu = \int_X f\chi_E \; d\mu$$

for any nonnegative measurable function f.

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[15] **5.** Guess the limit of

$$\int_0^n \left(1+\frac{x}{n}\right)^n e^{-2x}dx$$

as n tends to infinity. Prove that your guess is correct.

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[15] **6.** Let $X,\ Y,\ Z$ be nonempty sets and $\mathcal{L} \subset \mathcal{P}(X)$, $\mathcal{M} \subset \mathcal{P}(Y)$, $\mathcal{N} \subset \mathcal{P}(Z)$ be σ–algebras. Define $\mathcal{L} \otimes \mathcal{M} \otimes \mathcal{N} \subset \mathcal{P}(X \times Y \times Z)$ to be the σ–algebra generated by

$$\left\{\ A \times B \times C \ \middle|\ A \in \mathcal{L},\ B \in \mathcal{M},\ C \in \mathcal{N}\ \right\}$$

Prove that $(\mathcal{L} \otimes \mathcal{M}) \otimes \mathcal{N} = \mathcal{L} \otimes \mathcal{M} \otimes \mathcal{N}$.

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[15] **7.** For $j = 1, 2$, let μ_j and ν_j be σ–finite measures on $(X_j, \mathcal{M}_j)$ with $\nu_j \ll \mu_j$. Prove that $\nu_1 \times \nu_2 \ll \mu_1 \times \mu_2$ and

$$\tfrac{d(\nu_1 \times \nu_2)}{d(\mu_1 \times \mu_2)}(x_1, x_2) = \tfrac{d\nu_1}{d\mu_1}(x_1)\tfrac{d\nu_2}{d\mu_2}(x_2)$$

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Be sure that this examination has 13 pages including this cover

The University of British Columbia
Sessional Examinations - December 2006

Mathematics 420/507
Measure Theory and Integration

Closed book examination | Time: $2\frac{1}{2}$ hours

Name ______________________ **Signature** ______________________

Student Number______________________ **Instructor's Name** ______________________

Section Number ______________________

Special Instructions:

No calculators, notes, or other aids are allowed.

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2. Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.

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2		15
3		15
4		15
5		15
6		15
7		15
Total		100