

The University of British Columbia
Final Examination - April 11, 2013
Mathematics 419 Stochastic Processes

Time: 2.5 hours

Last Name _____ First _____ Signature _____

Student Number _____

Special Instructions:

No books, notes, or calculators. Marks depend on quality of supporting arguments. You can use parts of problems to prove other parts without having done the parts.

Rules governing examinations

- Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCcard for identification.
- Candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.
- No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no candidate shall be permitted to enter the examination room once the examination has begun.
- Candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.
- Candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
 - (a) speaking or communicating with other candidates, unless otherwise authorized;
 - (b) purposely exposing written papers to the view of other candidates or imaging devices;
 - (c) purposely viewing the written papers of other candidates;
 - (d) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
 - (e) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s)–(electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).
- Candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.
- Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.
- Candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
Total:	70	

When you want to use a theorem briefly state hypotheses. for example:
Thm. Irreducible $\Rightarrow \pi$ unique if π exists.

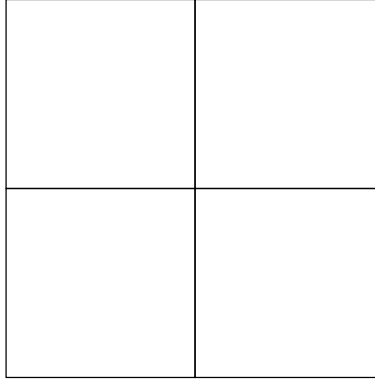


Figure 1

1. (10 points) Let $S = (S_n, n = 0, 1, 2, \dots)$ be a random walk on the state space V whose elements are the vertices in the graph in Figure 1. At each step the random walker moves to a nearest neighbour vertex chosen uniformly and independently.
 - (a) Find the stationary distribution π and write the probabilities next to the vertices in Figure 1.

(b) If the walker starts at the vertex c in the centre, what is the expected time to return?

(c) Does $\mathbb{P}(S_n = x)$ converge, as $n \rightarrow \infty$, to π_x for each $x \in V$?

(d) Does $\frac{1}{n} \sum_{m=1}^n \mathbb{1}_{S_m=x}$, as $n \rightarrow \infty$, converge to π_x for each $x \in V$?

2. (10 points) Let $X = (X_n, n = 0, 1, \dots)$ be a Markov chain with transition probability matrix P and state space S ; let A be a proper subset of S and let τ be the first time $n \geq 0$ such that $X_n \in A$. Suppose that $f = (f_i, i \in S)$ is a given bounded non-negative function and $h = (h_i, i \in S)$ solves

$$h_i = \begin{cases} f_i & i \in A \\ \sum_{j \in S} P_{ij} h_j & i \notin A. \end{cases} \quad (1)$$

- (a) Define the probability transition matrix P in terms of X .

- (b) Suppose T is the last time $n \geq 0$ such that $X_n \in A$. Is T a stopping time for X ?

(c) Suppose that $h = (h_i, i \in S)$ is a non-negative solution to (1). Show that, for $i \notin A$,

$$h_i = \mathbb{P}_i(h_{X_\tau} \mathbb{1}_{\tau \in \{1,2\}}) + \sum_{i_1, i_2 \in S \setminus A} P_{ii_1} P_{i_1 i_2} h_{i_2}. \quad (2)$$

(d) If $h \geq 0$ solves (1), is $h_i = \mathbb{E}_i(f_{X_\tau} \mathbb{1}_{\tau < \infty})$ always true, sometimes true or never true?

3. (10 points) Let X be random walk on the integers \mathbb{Z} which moves one step to the right with probability p and one step to the left with probability $q = 1 - p$.

(a) Let $\lambda = (\lambda_i, i \in \mathbb{Z})$ be a stationary distribution or more generally a stationary vector with nonnegative components. What equations express stationarity? Give the answer in terms of P and then in terms of p .

(b) One solution to part(a) is $\lambda_i = 1$. Find all solutions when $p \in [\frac{1}{2}, 1)$.

(c) What is a positively recurrent state? Let $p = \frac{1}{2}$. How can you deduce from your answer to part(b) implies that all states are not positively recurrent?

4. (10 points) At all times $n = 0, 1, \dots$, an urn contains N balls; some are black and the others are white. Let X_n be the number of white balls in the urn at time n . Independently of X_n a ball is chosen uniformly at random from the urn. Independently of X_n and the choice the chosen ball is replaced by a white ball with probability p and with a black ball with probability $q = 1 - p$, where $p \in (0, 1)$.
- (a) Find formulas in terms of N, i, p, q for the transition probabilities $p_{i,i+1}, p_{i,i}, p_{i,i-1}$ and check your answers (probabilities sum to one).

(b) Explain by a theorem why X has a unique stationary distribution π .

- (c) Find π and at the same time show that the chain is reversible. If you cannot do this for general N then do it for, e.g., $N = 3$.

5. (10 points) A particle in \mathbb{Z}^2 has position (X_t, Y_t) where X_t is a Poisson process of rate 1 jump per second and Y_t is an independent Poisson process of rate 2 jumps per second.
- (a) What is the probability that the particle makes exactly 5 jumps in the time interval $[0, 2]$. You can give your answer in a form like $\frac{1}{4!}e^4$.
- (b) What is the probability that the particle advances in the y direction by exactly 3 jumps before making any progress in the x direction?
- (c) (X_t, Y_t) is a continuous time Markov process. It has an infinitesimal generator G with matrix elements $g_{(x,y),(x',y')}$ where (x, y) and (x', y') are states in \mathbb{Z}^2 . What is $g_{(x,y),(x,y)}$?

6. (10 points) Let $S = (S_n, n = 0, 1, \dots)$ be symmetric simple random walk started with $S_0 = 100$.

(a) What does the supermartingale convergence theorem say?

(b) Let X_n be a martingale and let T be a stopping time that is almost surely finite. True or not always true: for $n = 0, 1, \dots$, $\mathbb{E}X_0 = \mathbb{E}X_{n \wedge T}$?

(c) S_n is a martingale. Does $\lim_{n \rightarrow \infty} S_n$ exist a.s.?

(d) Let T be the first time $n \geq 0$ such that $S_n = 0$. Does $\lim_{n \rightarrow \infty} S_{n \wedge T}$ exist?

(e) Give an example of a martingale X_n and a stopping time T which is almost surely finite such that $\mathbb{E}X_T \neq \mathbb{E}X_0$.

7. (10 points) Let $Y = (Y_n, n = 1, 2, \dots)$ be an i.i.d. sequence of random variables s.t. Y_1 equals one with probability $\frac{1}{2}$ and minus one with probability $\frac{1}{2}$. Define $M = (M_n, n = 0, 1, \dots)$ by $M_0 = 0$ and $M_n = M_{n-1}(Y_n + 1) + nY_n$ for $n > 0$. Define T to be the first time $n \geq 3$ such that $(Y_{n-2}, Y_{n-1}, Y_n) = (1, 1, 1)$.
- (a) Show that M_n is a martingale with respect to Y .

- (b) What does M_n equal when
- (i) $Y_n = -1$,
 - (ii) $(Y_{n-1}, Y_n) = (-1, 1)$,
 - (iii) $(Y_{n-2}, Y_{n-1}, Y_n) = (-1, 1, 1)$?

(c) Find ET .