The University of British Columbia

Final Examination - April 11, 2013

Mathematics 419 Stochastic Processes

Time: 2.5 hours

Last Name ______________________ **First** ______________ **Signature** ____________________

Student Number ______________________

Special Instructions:

No books, notes, or calculators. Marks depend on quality of supporting arguments. You can use parts of problems to prove other parts without having done the parts.

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
Total:	70	

When you want to use a theorem briefly state hypotheses. for example:
Thm. Irreducible $\Rightarrow \pi$ unique if π exists.

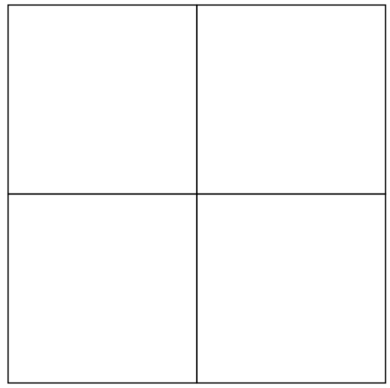

Figure 1

1. (10 points) Let $S = (S_n, n = 0, 1, 2, \dots)$ be a random walk on the state space V whose elements are the vertices in the graph in Figure 1. At each step the random walker moves to a nearest neighbour vertex chosen uniformly and independently.
   (a) Find the stationary distribution π and write the probabilities next to the vertices in Figure 1.

   (b) If the walker starts at the vertex c in the centre, what is the expected time to return?

(c) Does $\mathbb{P}(S_n = x)$ converge, as $n \to \infty$, to π_x for each $x \in V$?

(d) Does $\frac{1}{n} \sum_{m=1}^{n} \mathbb{1}_{S_m = x}$, as $n \to \infty$, converge to π_x for each $x \in V$?

2. (10 points) Let $X = (X_n, n = 0, 1, \dots)$ be a Markov chain with transition probability matrix P and state space S; let A be a proper subset of S and let τ be the first time $n \geq 0$ such that $X_n \in A$. Suppose that $f = (f_i,\ i \in S)$ is a given bounded non-negative function and $h = (h_i,\ i \in S)$ solves

$$h_i = \begin{cases} f_i & i \in A \\ \sum_{j \in S} P_{ij} h_j & i \notin A. \end{cases} \tag{1}$$

(a) Define the probability transition matrix P in terms of X.

(b) Suppose T is the last time $n \geq 0$ such that $X_n \in A$. Is T a stopping time for X?

(c) Suppose that $h = (h_i, i \in S)$ is a non-negative solution to (1). Show that, for $i \notin A$,

$$h_i = \mathbb{P}_i(h_{X_\tau} \mathbb{1}_{\tau \in \{1,2\}}) + \sum_{i_1, i_2 \in S \setminus A} P_{ii_1} P_{i_1 i_2} h_{i_2}. \tag{2}$$

(d) If $h \geq 0$ solves (1), is $h_i = \mathbb{E}_i\big(f_{X_\tau} \mathbb{1}_{\tau < \infty}\big)$ always true, sometimes true or never true?

3. (10 points) Let X be random walk on the integers $\mathbb{Z}$ which moves one step to the right with probability p and one step to the left with probability $q = 1 - p$.

   (a) Let $\lambda = (\lambda_i, i \in \mathbb{Z})$ be a stationary distribution or more generally a stationary vector with nonnegative components. What equations express stationarity? Give the answer in terms of P and then in terms of p.

   (b) One solution to part(a) is $\lambda_i = 1$. Find all solutions when $p \in [\frac{1}{2}, 1)$.

   (c) What is a positively recurrent state? Let $p = \frac{1}{2}$. How can you deduce from your answer to part(b) implies that all states are not positively recurrent?

4. (10 points) At all times $n = 0, 1, \ldots,$ an urn contains N balls; some are black and the others are white. Let X_n be the number of white balls in the urn at time n. Independently of X_n a ball is chosen uniformly at random from the urn. Independently of X_n and the choice the chosen ball is replaced by a white ball with probability p and with a black ball with probability $q = 1 - p$, where $p \in (0, 1)$.

   (a) Find formulas in terms of N, i, p, q for the transition probabilities $p_{i,i+1}, p_{i,i}, p_{i,i-1}$ and check your answers (probabilities sum to one).

   (b) Explain by a theorem why X has a unique stationary distribution π.

(c) Find π and at the same time show that the chain is reversible. If you cannot do this for general N then do it for, e.g., $N = 3$.

5. (10 points) A particle in $\mathbb{Z}^2$ has position (X_t, Y_t) where X_t is a Poisson process of rate 1 jump per second and Y_t is an independent Poisson process of rate 2 jumps per second.

   (a) What is the probability that the particle makes exactly 5 jumps in the time interval $[0, 2]$. You can give your answer in a form like $\frac{1}{4!}e^4$.

   (b) What is the probability that the particle advances in the y direction by exactly 3 jumps before making any progress in the x direction?

   (c) (X_t, Y_t) is a continuous time Markov process. It has an infinitesmal generator G with matrix elements $g_{(x,y),(x',y')}$ where (x, y) and (x', y') are states in $\mathbb{Z}^2$. What is $g_{(x,y),(x,y)}$?

6. (10 points) Let $S = (S_n, n = 0, 1, \dots)$ be symmetric simple random walk started with $S_0 = 100$.

   (a) What does the supermartingale convergence theorem say?

   (b) Let X_n be a martingale and let T be a stopping time that is almost surely finite. True or not always true: for $n = 0, 1, \dots$, $\mathbb{E}X_0 = \mathbb{E}X_{n\wedge T}$?

   (c) S_n is a martingale. Does $\lim_{n\to\infty} S_n$ exist a.s.?

   (d) Let T be the first time $n \geq 0$ such that $S_n = 0$. Does $\lim_{n\to\infty} S_{n\wedge T}$ exist?

   (e) Give an example of a martingale X_n and a stopping time T which is almost surely finite such that $\mathbb{E}X_T \neq \mathbb{E}X_0$.

7. (10 points) Let $Y = (Y_n,\, n = 1, 2, \dots)$ be an i.i.d. sequence of random variables s.t. Y_1 equals one with probability $\frac{1}{2}$ and minus one with probability $\frac{1}{2}$. Define $M = (M_n, n = 0, 1, \dots)$ by $M_0 = 0$ and $M_n = M_{n-1}(Y_n + 1) + nY_n$ for $n > 0$. Define T to be the first time $n \geq 3$ such that $\big(Y_{n-2}, Y_{n-1}, Y_n\big) = (1, 1, 1)$.

   (a) Show that M_n is a martingale with respect to Y.

   (b) What does M_n equal when

      (i) $Y_n = -1$,

      (ii) $\big(Y_{n-1}, Y_n\big) = (-1, 1)$,

      (iii) $\big(Y_{n-2}, Y_{n-1}, Y_n\big) = (-1, 1, 1)$?

(c) Find $\mathbb{E}T$.