Math 419 Final Exam April 2012

Section: 201 Instructor: Ed Perkins Duration: 2.5 hours .

Instructions:

- Write your name and student ID on the front page.
- This examination contains five questions worth a total of 104 points.
- Write each answer **very clearly** below the corresponding question (Use back of page if needed).
- No calculators, books, notebooks or any other written materials are allowed.
- **Good luck!**

1. (22 points) Consider the Markov Chain with state space $\{1, 2, 3, 4\}$ and transition matrix

$$P = \begin{pmatrix} .5 & .2 & .25 & .05 \\ 0 & .5 & .5 & 0 \\ 0 & 0 & .25 & .75 \\ 0 & 0 & .25 & .75 \end{pmatrix}.$$

(a) Find the communicating classes and classify each as transient, positive recurrent or null recurrent. You need not justify your answer.

(b) Find the period of each state. Justify your answers.

(c) Show the restriction of P to the state space $\{3, 4\}$ describes an irreducible aperiodic Markov Chain. Find its stationary distribution π.

(d) Find $\lim_{n\to\infty} P^n$.

2. (15 points) (a) State carefully: the Martingale Convergence Theorem.

(b) Define carefully

i. A uniformly integrable collection of random variables $\{X_i : i \in I\}$.

ii. The previsible square function $(\langle M\rangle_n : n \in \mathbb{Z}_+)$ of an $(\mathcal{F}_n)$-martingale $(M_n : n \in \mathbb{Z}_+)$.

(c) True or False. If True, give a proof; if False, provide a counter-example. If $(M_n : n \in \mathbb{Z}_+)$ is a non-negative martingale satisfying $M_0 = 0$ and $E(\langle M\rangle_\infty) < \infty$, then M_n converges a.s. and in L^1 to an integrable limit. Here $\langle M\rangle_\infty = \lim_{n\to\infty}\langle M\rangle_n$.

3. (15 points) Let $\mathcal{G}$ be a sub-σ-field of $\mathcal{F}$.

(a) Define carefully: The conditional expectation $E(X|\mathcal{G})$ of an integrable random variable X on a probability space $(\Omega, \mathcal{F}, P)$ given $\mathcal{G}$.

(b) Assume X is a non-negative r.v. which is not integrable.

(i) Prove that $E(X \wedge n|\mathcal{G})$ converges a.s. to a limit r.v. taking values in $[0, \infty]$.

(ii) Define $E(X|\mathcal{G}) = \limsup_{n\to\infty} E(X \wedge n|\mathcal{G})$. Show that $E(X|\mathcal{G})$ satisfies the same defining properties (in (a) above) as the conditional expectation of an integrable r.v. X.

(iii) Give an example of an X and $\mathcal{G}$ as above where $E(X|\mathcal{G}) < \infty$ a.s. but $E(X) = \infty$.

4. (30 points) (a) Assume $\{Z_n : n \in \mathbb{N}\}$ are non-negative iid random variables with mean one, $\mathcal{F}_n = \sigma(Z_1, \ldots, Z_n)$, and set $M_n = \prod_{i=1}^n Z_i$ (so $M_0 = 1$). Show that M_n is an $(\mathcal{F}_n)$-martingale and converges a.s. as $n \to \infty$.

(b) Assume $\{X_n\}$ are iid random variables such that $P(X_n = 1) = P(X_n = -1) = \frac{1}{2}$ and set $S_n = \sum_{i=1}^n X_i$, $\mathcal{F}_n = \sigma(S_1, \ldots, S_n)$, and $T = \min\{n \geq 0 : S_n = 1\}$, where $\min \emptyset = \infty$.

(i) Show, using the definition of stopping time, that T is an $(\mathcal{F}_n)$-stopping time.

(ii) Show that $T < \infty$ a.s. (Hint: This is easy if you think of Markov Chain theory).

(iii) If $\theta \geq 0$ and $\cosh\theta = (e^\theta + e^{-\theta})/2$, show that $M_n = \frac{e^{\theta S_n}}{(\cosh\theta)^n}$ is an $(\mathcal{F}_n)$-martingale.

(iv) Show that $E(M_{T\wedge n}) = 1$ for all n. Now show that $E(M_T) = 1$.

(v) Use the above to prove that $E(x^T) = x^{-1}[1 - \sqrt{1-x^2}]$ for $0 \leq x \leq 1$].

(vi) Prove that $E(T) = \infty$. Justify this conclusion carefully.

5. (22 points) Let B be a standard Brownian motion and define $M_t = \sup_{0\leq s\leq t} B_s$. NOTE: DO NOT USE THE LAW OF THE ITERATED LOGARITHM IN THIS QUESTION–UNLESS YOU GIVE A COMPLETE PROOF WHICH IS NOT RECOMMENDED.

(a) What is the cumulative distribution function of M_t (you need not justify your answer because this is a restatement of the Reflection Principle)?

(b) State the Borel-Cantelli Lemma.

(c) Let $0 < p < \frac{1}{2}$. Prove that

$$\limsup_{n\to\infty} \frac{M_{1/n}}{n^{-p}} = 0 \quad a.s.$$

(d) Prove that for p as above,

$$\limsup_{t\downarrow 0} \frac{|B_t|}{t^p} = 0 \ a.s.$$

(e) Prove that for $p > \frac{1}{2}$,

$$\limsup_{t\downarrow 0} \frac{|B_t|}{t^p} = \infty \ a.s.$$

Hint: One approach is to modify the proof of nowhere differentiability of the Brownian path.