1. [15/100] A rare inherited genetic disease behaves as follows. For people with the gene, no symptoms arise at all with probability $1 - p$, and symptoms appear with probability p at age 50. Charles is 49. His father Arthur (who is 80) and his sister Belinda (who is 52) have no symptoms. However Arthur's father Zoltan did have symptoms.

Because the disease is rare, you can neglect the possibility that Zoltan had 2 copies of the gene, or that Arthur's wife or mother had any copies of the gene. People without the gene never exhibit symptoms.

What is the probability that Charles will exhibit symptoms next year?

2. [13/100] Let X_i be i.i.d $\text{Ber}(p)$ r.v., and N be Poisson with parameter λ and independent of the X_i. Set

$$S = \sum_{i=1}^{N} X_i,$$

and let $Y = N - S$.
(a) Calculate the joint distribution of S and Y.

(b) Find $E(N|S)$.

3. [15/100] Let $(S_n, n \geq 0)$ be the symmetric SRW on $\mathbb{Z}$ (so $S_n = \sum_{i=1}^n X_i$ where X_i are i.i.d and $P(X_i = -1) = P(X_i = 1) = 1/2$). Let $N > 0$, and A_N be the event that S hits N before 0.

(a) If $0 \leq k \leq N$ find $P(A_N | S_0 = k)$.

(b) Find $P(S_1 = k + 1 | S_0 = k, A_N)$.

4. [15/100] (a) Let X and Y be jointly continuous random variables. Define $E(X|Y)$.

(b) Show that $E\big(E(X|Y)\big) = E(X)$.

(c) Let X have an exponential distribution with mean λ. Let Y be a r.v. which has an exponential distribution with mean X. What is the joint p.d.f. of X, Y?

(d) Find $E(Y)$.

5. [15/100] True or False. If True, give a brief proof. If False, provide a counter-example.
(a) If events A, B are independent then A and B^c are independent.

(b) If $X_n \to C$ in probability, where C is a constant, then $X^2 \to C^2$ in probability.

(c) If $\varphi(t)$ is the characteristic function of a r.v. X then

$$|\varphi(t+h) - \varphi(t)| \le |\varphi(h) - 1|$$

for all $t \in \mathbb{R}$ and $|h| < 1$.

6. [13/100] (a) Define *convergence in distribution* of a sequence of random variables X_n.

(b) Let X_n and Y_n be independent r.v. with a $\mathrm{Bin}(n, \lambda/n)$ distribution. Let $Z_n = X_n - Y_n$. Prove that (Z_n) converge in distribution. (State carefully any general theorems you use.) What can you say about the limiting distribution?

7. [14/100] An aircraft seats $N = 400$ passengers. Passengers have weights with mean 90 kg, and variance 300 kg^2. Passengers have no luggage with probability 1/2, and have luggage with weight 20 kg with probability 1/2. (All these may be assumed to be independent.) Let W be the total weight of passengers and baggage. The airline wants to find x such that $P(W > x) = 0.01$. Use the central limit theorem to find an approximation to x.

If $\Phi(y)$ is the distribtion function of a $N(0, 1)$ r.v. then $\Phi(2.33) \simeq 0.99$.