

Math 412, Spring Term 2014

Final Exam

April 16th, 2014

Do not turn this page over until instructed

- On the top of each exam booklet include the serial number of this paper. If you use more than one booklet, number them.

Instructions

- You will have 180 minutes for this exam.
- You may not use any books, notes or electronic devices.
- Solutions should be written clearly, in complete English sentences, showing all your work.
- If you use a result from the lectures or the problem sets, quote it properly.
- Good luck!

Student Conduct during Examinations

1. Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBC card for identification.
2. Examination candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.
3. No examination candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no examination candidate shall be permitted to enter the examination room once the examination has begun.
4. Examination candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.
5. Examination candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
 - (a) speaking or communicating with other examination candidates, unless otherwise authorized;
 - (b) purposely exposing written papers to the view of other examination candidates or imaging devices;
 - (c) purposely viewing the written papers of other examination candidates;
 - (d) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
 - (e) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s)—(electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).
6. Examination candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.
7. Examination candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

1. (10 pts) In this problem (but not later ones) a brief justification will suffice.
 - (a) A real matrix has the characteristic polynomial $(x-1)(x-3)^3$ and the minimal polynomial $(x-1)(x-3)^2$. What is its Jordan form?
 - (b) Find the Jordan form of the square of that matrix.
2. (25 pts) We will solve the differential equation $u'' = 2u' - u$.
 - (a) Set $\underline{v} = \begin{pmatrix} u \\ u' \end{pmatrix}$. Find a matrix $A \in M_2(\mathbb{R})$ such $\underline{v}' = A\underline{v}$.
 - (b) Find an invertible matrix S and a matrix J in Jordan form such that $A = SJS^{-1}$.
 - (c) Evaluate $\exp(At)$.
3. (15 pts) Let V_1, V_2 be two F -vector spaces, and let $N_i \in \text{End}_F(V_i)$ be linear maps.
 - (a) Construct the linear map $N_1 \oplus N_2 \in \text{End}_F(V_1 \oplus V_2)$.
 - (b) Show that $N_1 \oplus N_2$ is nilpotent iff both of N_1, N_2 are.
4. (20 pts) Let V be an n -dimensional F -vector space ($\text{char}(F) \neq 2$), and let (\cdot, \cdot) be a non-zero symmetric bilinear form on V .
 - (a) Use the identity $(\underline{u} + \underline{v}, \underline{u} + \underline{v}) = (\underline{u}, \underline{u}) + 2(\underline{u}, \underline{v}) + (\underline{v}, \underline{v})$ to find $\underline{v} \in V$ such that $(\underline{v}, \underline{v}) \neq 0$.
 - (b) Find a basis $\{\underline{v}_i\}_{i=1}^n \subset V$ such that $(\underline{v}_i, \underline{v}_j) = a_i \delta_{ij}$ for some $a_i \in F$.
5. (15 pts) Let V be a complete normed space. Let $A, B \in \text{End}_b(V)$ be bounded linear maps from V to itself such that A, B commute. Show that:
 - (a) If $\|B\|$ is small enough (depending on A), then $C = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \exp(-kA)B^k$ converges absolutely.
 - (b) Under the same condition, $\exp(A + C) = \exp(A) + B$.
6. (15 pts) Suppose that $g \in \text{GL}_n(\mathbb{C})$ has $g^k = \text{Id}$ for some $k \geq 2$. Show that g is diagonalizable.