The University of British Columbia

Final Examination - Dec 9, 2014

Mathematics 405/607

All Sections

Closed book examination. No calculators. Time: 2.5 hours

Special Instructions:

No books, notes, or calculators are allowed. Show all your work, little or no credit will be given for a numerical answer without the correct accompanying work. Write your answers in booklets provided.

Rules governing examinations

- Each candidate must be prepared to produce, upon request, a UBCcard for identification.
- Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.
- No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination.
- Candidates suspected of any of the following, or similar, dishonest practises shall be immediately dismissed from the examination and shall be liable to disciplinary action.
  (a) Having at the place of writing any books, papers or memoranda, calculators, computers, sound or image players/recorders/transmitters (including telephones), or other memory aid devices, other than those authorized by the examiners.
  (b) Speaking or communicating with other candidates.
  (c) Purposely exposing written papers to the view of other candidates or imaging devices. The plea of accident or forgetfulness shall not be received.
- Candidates must not destroy or mutilate any examination material; must hand in all examination papers; and must not take any examination material from the examination room without permission of the invigilator.
- Candidates must follow any additional examination rules or directions communicated by the instructor or invigilator.

Part A. Do all 9 questions. 5 marks each.

A1: Answer the following questions with a brief (one or two sentence) explanation of your reasoning:

(a) [3 marks] A numerical method for ODE boundary value problems is tested with a known solution using uniform step sizes h. The errors for several values of h are given below. What is the convergence order of the method?

h	Error
0.1	0.123
0.05	0.0156
0.025	0.00197

(b) [2] The solution of a nonlinear system is being found iteratively using Newton's method. A test problem with an exact solution is used to investigate the method. Errors after several iterations are shown in the table below. Is the method working properly?

iteration	Error
4	0.895
5	0.448
6	0.224

A2: Consider the second order finite difference approximation of the second derivative:

$$D_2 U_j := \frac{1}{h^2}(U_{j+1} - 2U_j + U_{j-1})$$

where U_j are N values on a grid of the unit interval with equal spacing h ($h = 1/N$). Use von Neumann analysis to find the eigenvalues of D_2 when U_j values are taken to be N-periodic. *Note:* You may have memorized this result, but show details of the derivation for marks on this question.

A3: Consider a reaction diffusion problem for $u(x,t)$ with u 1-periodic in x. It is discretized with a finite difference method in space with spatial steps h and with backward Euler method in time with steps k:

$$U_j^{n+1} = U_j^n + kD_2 U_j^{n+1} + kf(U_j^{n+1})$$

where $U_j^n \approx u(jh, nk)$ and $f(u)$ is a given function. $\mathbf{U}^{n+1}$ is vector with N components ($N = 1/h$). At every time step, the equation above is a nonlinear system for $\mathbf{U}^{n+1}$. Write out the entries of the Jacobian matrix for this system. *Note:* some entries will involve values of f'.

The questions **A4**, **A5** and **A6** below concern the following approximation of 1-periodic in space solutions $u(x,t)$ of the one-way wave equation $u_t + u_x = 0$:

$$U_j^{n+1} = U_j^n - \frac{k}{h}(U_j^n - U_{j-1}^n).$$

Initial data $\mathbf{U}^0$ are given. Time steps are taken proportional to space steps:

$$k = Ch$$

with $C = 0.9$.

A4: Show that the scheme satisfies a maximum principle, that is that

$$\max_j U_j^n \leq \max_j U_j^0$$

for every $n > 0$.

A5: What is the order of the truncation error of the scheme?

A6: Consider the modified scheme

$$U_j^{n+1} = U_j^n - \frac{k}{h}(U_{j+1}^n - U_j^n).$$

with the same initial data and time steps. Use von Neumann analysis to show that this scheme is not stable.

A7: Consider the following time stepping method for the problem $\dot{u} = f(u,t)$:

$$U^{n+1} = U^n + \frac{k}{2}\left(3f(U^n, nk) - f(U^{n-1}, (n-1)k)\right)$$

Identify the order of the truncation error of the method. Other aspects of this method are considered in problem B1.

A8: Consider approximating the function $f(x)$ on the interval [-1,1] using a linear function $L(x) = Ax + B$ that agrees with f at two distinct points x_1 and x_2 in the interval, that is $L(x_1) = f(x_1)$ and $L(x_2) = f(x_2)$. Show that

$$|L(x) - f(x)| \leq CK_2$$

where

$$K_2 = \max_{x \in [-1,1]} |f''|$$

and C is a constant that depends on the choice of x_1 and x_2. Other aspects of this approximation are considered in problem B2.

A9: Consider the following problem for scalar functions $x(t)$ and $y(t)$:

$$\frac{dx}{dt} = f(x, y) \quad (1)$$

$$g(x, y) = 0 \quad (2)$$

where the functions f and g are given. Initial values $x(0)$ and $y(0)$ are also given, that satisfy equation (2).

(a) [3 marks] Describe a consistent numerical method for time stepping approximate solutions $X^n \approx x(nk)$ and $Y^n \approx y(nk)$, where k is given time step. It is desirable that (2) be satisfied by the approximation, that is

$$g(X^n, Y^n) = 0$$

to high accuracy (machine precision) at each n.

(b) [2] Describe how you would test your method above. Give enough detail that someone could implement your test.

Part B. Do *one* of the following three problems. 5 marks

B1: Consider the time stepping method from A7, for approximating $\dot{u} = f(u,t)$:

$$U^{n+1} = U^n + \frac{k}{2}\left(3f(U^n, nk) - f(U^{n-1}, (n-1)k)\right).$$

Determine what points (if any) on the negative real axis are in the stability region of the method.

B2: Consider approximating the function $f(x)$ on the interval [-1,1] using a linear function $L(x) = Ax + B$ that agrees with f at two distinct points x_1 and x_2 in the interval as considered in A8. It was shown that

$$|L(x) - f(x)| \leq CK_2$$

where

$$K_2 = \max_{x\in[-1,1]} |f''|$$

and C is a constant that depends on the choice of x_1 and x_2. Identify the values of x_1 and x_2 that make the constant C above as small as possible.

B3: Consider the following equation for $u(x,t)$ defined on the real line $x \in (-\infty, \infty)$:

$$u_{tt} - u_t = u_{xx} + f(x,t)$$

where the functions $f(x,t)$, $u(x,0)$ and $u_t(x,0)$ are given. These functions are zero outside the x-interval [-1,1]. The solution satisfies (for each t):

$$\lim_{|x|\to\infty} u(x,t) = 0$$

(a) [3 marks] Describe a numerical scheme to approximate the solution of this problem. Include details on how you will handle the infinite domain aspect of it.

(b) [2] Consider your scheme in the periodic domain setting. Use von Neumann analysis to determine the time step restrictions your method will have (if any).