MATH 405/607E Numerical methods for differential equations

Final Exam, December 10th 2013

1. **Interpolation**. Find a natural cubic spline interpolant, $s(x)$, for a function $\sin(x)$ over the interval $x \in (0, \pi)$. You are given three data points $x_k = \{0, \pi/2, \pi\}$ and corresponding three function values $\sin(x_k) = \{0, 1, 0\}$. Why natural spline is a good choice? Once you find the spline function, evaluate it at $x = \pi/4$, and compare the answer to $\sin(\pi/4) = 1/\sqrt{2}$. Estimate the accuracy of the spline approximation by calculating m in $|s(\pi/4) - 1/\sqrt{2}| = O(10^{-m})$. Hint: $a^2 - b^2 = (a-b)(a+b)$, $\sqrt{2} \approx 1.4$.

2. **Numerical integration**. Use the fact that Gauss-Legendre quadrature with N points allows to calculate the integrals of polynomials of degree $2N-1$ exactly to find the locations and weights for $N=2$. Find the formula for calculating the n dimensional integral over n dimensional hypercube (i.e. n integrals with identical -1 to 1 limits) using N Gauss points. Hint: use 1D integration recursively.

3. **Initial value ODE**. The leapfrog method for solving ODE $y' = f(x, y)$ can be formulated as
$$\frac{y_{n+1} - y_{n-1}}{2h} = f(x_n, y_n),$$
where $x_n = hn$, h is the step size, while y_n is an approximation for $y(x_n)$. Find the order of the truncation error and the stability region for this scheme. Are there any potential issues regarding the implementation of the scheme? How would you solve them? Hints: i) if $|G| = 1$, then $G = e^{i\phi}$, ii) given y_0, try to find y_1 and y_2.

4. **BVP**. Consider the following boundary value problem (BVP)
$$u_{xxxx} = f(x), \quad u(0) = 0, \quad u_x(0) = 0, \quad u_{xx}(1) = 0, \quad u_{xxx}(1) = 0.$$
Use the central difference to discretize the differential equation. Use forward difference to deal with the left boundary condition (at $x=0$), while use backward differences to approximate the right boundary conditions (at $x=1$). In this case, you do not need to introduce pseudo mesh points. Try to use the simplest forward/backward differences to avoid unnecessary complexity. Write the system of algebraic equations in the matrix form. If you solve the system of equations (for any given $f(x)$), and then compare the result to the exact solution, how would the error (maximum absolute difference) depend on the mesh size h (i.e. $O(h^m)$, find m)? Hint: due to the symmetry, the truncation error for the central difference can have only even powers of h. Also, $\delta^4 u_n = \delta^2(\delta^2 u_n)$, where $\delta u_n = u_{n+1/2} - u_{n-1/2}$.

5. **Von Neumann stability analysis**. Discretize
$$\frac{\partial u}{\partial t} = -\frac{\partial^4 u}{\partial x^4}$$
using central difference in space and forward difference in time. Find the condition for the time step which ensures the stability of the numerical solution (e.g. for the forward Euler scheme for the diffusion equation $\Delta t < \frac{1}{2}\Delta x^2$).