This examination has 5 questions on 2 pages.

The University of British Columbia
Final Examinations—December 2005

Mathematics 402
Calculus of Variations (*Professor Loewen*)

Open book examination. Time: $2\frac{1}{2}$ hours
Any written materials are allowed. No electronic devices.

[20] **1.** Use $L(t, x, v) = \sqrt{x(1 + v^2)}$ to solve parts (a)–(c) below:

(a) Find all extremals $x(\cdot)$ of L that obey $x(0) = 1$ and $x(2) = 5$.

(b) Each extremal in part (a), together with the interval $[0, 2]$, satisfies the first-order necessary conditions for optimality in a point-to-curve problem of this form:

$$\min_{\substack{b>0,\\ x\in PWS[0,b]}} \left\{ \int_0^b \sqrt{x(t)\,(1 + \dot{x}(t)^2)}\, dt \;:\; x(0) = 1,\; x(b) = 5 + m(b - 2) \right\}.$$

Find the slope m (a constant) corresponding to each extremal.

(c) For each extremal in item (b), use second-order tests to support the strongest conclusions you can justify concerning optimality or non-optimality.

[20] **2.** Choose $b > 0$ and $x\colon [0, b] \to \mathbb{R}$ to minimize

$$\Lambda[x; b] = \int_0^b x(t)^2 \dot{x}(t)^2\, dt$$

subject to the endpoint restrictions $x(0) = 1$ and $x(b) = \sqrt{4 + b^2}$.

[20] **3.** Consider this isoperimetric problem on a fixed interval, assuming $\gamma > 0$:

$$\min_x \left\{ \int_0^2 \sqrt{1 + \dot{x}(t)^2}\, dt \;:\; x(0) = 0 = x(2),\; \int_0^2 x(t)\, dt = \gamma \right\}.$$

(a) Prove: If a minimizing arc exists, its graph must be an arc of an ellipse in the (t, x)-plane.

(b) Prove: When $\gamma = \dfrac{\pi}{2} - 1$, a minimizing arc does exist, and indeed its graph is an arc of a circle in the (t, x)-plane.
[Clue: *Proving* a given statement is often easier than *deriving* it.]

[20] **4.** Find the global minimizer, $\widehat{x}$, in the problem below. Show that $\widehat{x}$ is C^1 but not C^2.

$$\min\left\{\int_{-2}^{2}\left(\dot{x}(t)^4+3x(t)^2\right)\,dt\;:\;x(-2)=-1,\;x(2)=1\right\}.$$

Clues: Symmetry suggests that $\widehat{x}$ should be an odd function. Every odd function in C^1 has predictable behaviour at the origin, and this will help exploit WE2. Proving global optimality is part of the challenge here.

[20] **5.** Use these definitions in parts (a)–(c) below:

$$L(t,x,v)=\frac{v^2}{2}-\frac{x^2}{t(1-t)},$$

$$V(T,X)\stackrel{\text{def}}{=}\min_{x\in PWS[0,T]}\left\{\int_0^T L(t,x(t),\dot{x}(t))\,dt\;:\;x(0)=0,\;x(T)=X\right\}.$$

(a) Prove: On any subinterval $[a,b]$ of $(0,1)$, each arc $x(t;\alpha)=\alpha t(1-t)$, $\alpha\in\mathbb{R}$, is optimal relative to its endpoints for L. [Prove strong local minimality.]

(b) Use the information in (a) to make a "guess" at the minimum value $V(T,X)$ defined above. Call your guess $W(T,X)$.

(c) Find the Hamiltonian H corresponding to L, and write the Hamilton-Jacobi PDE for an unknown function $W=W(t,x)$.

(d) Evaluate $V(T,X)$ for $0<T<1$ and $X\in\mathbb{R}$. (Clue: prove that the "guess" in (b) is correct.)

(e) Prove that for every piecewise smooth $x\colon[0,1]\to\mathbb{R}$ with $x(0)=0=x(1)$,

$$\int_0^1\left[\frac{\dot{x}(t)^2}{2}-\frac{x(t)^2}{t(1-t)}\right]dt\geq 0.$$