MATH 401 FINAL EXAM – April 18, 2011

No notes or calculators allowed. Time: 2.5 hours. Total: 50 pts.

1. Consider the ODE problem for $u(x)$:

$$\begin{cases} Lu := (p(x)u')' + q(x)u = f(x), \quad 1 < x < 2 \\ u(1) = 2, \quad u(2) = 5 \end{cases} \tag{1}$$

($p(x)$, $q(x)$, and $f(x)$ are given functions).

(a) (2 pts.) Write down the problem satisfied by the Green's function $G_x(y) = G(x;y)$ for problem (1).

(b) (2 pts.) Express the solution $u(x)$ of (1) in terms of the Green's function $G_x(y)$.

(c) (3 pts.) If $p(x) = 1/x^2$ and $q(x) = 2/x^4$, find the Green's function $G(x;y)$ for (1).

(d) (3 pts.) Again with $p(x) = 1/x^2$ and $q(x) = 2/x^4$, find the solvability condition on $f(x)$ for (1) if the boundary conditions are changed to $u'(1) = 2$, $u(2) = 5$.

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2. The free-space Green's function for Δ in the plane $\mathbb{R}^2$ is $G_x^{\mathbb{R}^2}(y) = A \ln |y - x|$.

(a) (3 pts.) Derive the value of the constant A.

(b) (5 pts.) Use the method of images to find the Green's function for the following boundary-value problem for Laplace's equation in the quarter-plane,

$$\begin{cases} \Delta u = 0, & x_1 > 0,\ x_2 > 0 \\ u(0, x_2) = 0, & u(x_1, 0) = g(x_1) \end{cases}, \tag{2}$$

and find the solution $u(x_1, x_2)$.

(c) (2 pts.) Prove that problem (2) cannot have more than one solution $u(x)$ which decays at infinity (i.e. $\lim_{|x|\to\infty} u(x) = 0$).

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3. Let D be a bounded domain in $\mathbb{R}^n$, and let $\{\phi_j(x)\}_{j=1}^{\infty}$ be a complete, orthonormal set of eigenfunctions for $-\Delta$ on D with zero (Dirichlet) BCs: $-\Delta\phi_j(x) = \lambda_j\phi_j(x)$, $\lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots$.

(a) (4 pts.) Write down the problem satisfied by the Green's function $G(y, \tau; x, t)$ for the following problem for a heat-type equation

$$\begin{cases} u_t = \Delta u + u & x \in D,\ t > 0 \\ u(x,0) = u_0(x) & x \in D \\ u(x,t) \equiv 0 & x \in \partial D \end{cases},$$

and express the solution $u(x,t)$ in terms of the Greens function.

(b) (4 pts.) Find the Green's function as an eigenfunction expansion.

(c) (2 pts.) Under what condition will typical solutions **grow** with time?

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4. Consider the variational problem

$$\min_{u \in C^4([0,1])} \int_0^1 \left\{ \frac{1}{2}(u''(x))^2 + \frac{1}{2}(u(x))^2 - e^x u(x) \right\} dx.$$

(a) (5 pts.) Determine the problem (Euler-Lagrange equation plus BCs) that a minimizer $u(x)$ solves.

(b) (5 pts.) Find an approximate minimizer, using a Rayleigh-Ritz approach, with two trial functions $v_1(x) = e^x$, $v_2(x) = e^{-x}$.

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5. Let D denote the **half** unit disk:

$$D = \{\ (x_1,\ x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 < 1,\ x_2 > 0\ \},$$

and let λ_1 denote the first Dirichlet (zero BCs) eigenvalue of $-\Delta$ on D.

(a) (4 pts.) Find the best upper- and lower-bounds you can for λ_1 by comparing D with appropriate rectangles.

(b) (3 pts.) Write the variational principle for λ_1, and use it with trial function $v(x) = x_2(1 - x_1^2 - x_2^2)$ to find an upper bound for λ_1 (you may wish to compute in polar coordinates).

(c) (3 pts.) Find the exact value of λ_1 in terms of the first positive root of the Bessel function $J_1(z)$ (which is the solution of the ODE $z^2 J''(z) + zJ'(z) + (z^2 - 1)J(z) = 0$ which is finite at $z = 0$).

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