Be sure that this examination has 4 pages including this cover

The University of British Columbia

Sessional Examinations - April 2007

Mathematics 401

Green's Functions and Variational Methods

Closed book examination — Time: 2.5 hours

Special Instructions:

Do any 8 of 10 questions. If more than 8 questions are attempted, the best 8 marks will be taken. Each question is out of 10.
No notes, calculators, or books.

Rules governing examinations

1. Each candidate should be prepared to produce his or her library/AMS card upon request.
2. Read and observe the following rules:

No candidate shall be permitted to enter the examination room after the expiration of one half hour, or to leave during the first half hour of the examination.

Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.

CAUTION - Candidates guilty of any of the following or similar practices shall be immediately dismissed from the examination and shall be liable to disciplinary action.

(a) Making use of any books, papers or memoranda, other than those authorized by the examiners.

(b) Speaking or communicating with other candidates.

(c) Purposely exposing written papers to the view of other candidates. The plea of accident or forgetfulness shall not be received.

3. Smoking is not permitted during examinations.

[10] **1.** Consider the ODE BVP for $u(x)$, $x \in [0,1]$ given below:

$$u'' + xu' - u = f(x), \qquad u(0) = 0, u(1) = 0.$$

(a) [5 marks] Find the adjoint of this problem.

(b) [5] Let $G(s,x)$ be the Green's function for the original problem above, i.e.

$$u(x) = \int_0^1 G(s,x) f(s) ds$$

Write the *conditions* that $G(s,x)$ must satisfy. It is not necessary to find G explicitly.

[10] **2.** Determine which of the following problems for $u(x,t)$, $x \in \mathbf{R}$, $t \geq 0$ are well posed. Justify.

(a) [5] $u_{tt} = u_{xxxx}$ with $u(x,0)$ and $u_t(x,0)$ given.

(b) [5] $u_t = -u_{xx} - u_{xxxx}$ with $u(x,0)$ given.

[10] **3.** Consider $u(x,y)$ that solves

$$\Delta u = f(x,y)$$

with f given with compact support Ω contained in the unit disk centred at the origin. Recall that the solution can be written

$$u(x,y) = \frac{1}{2\pi} \int_\Omega \ln |(x,y) - (s,t)| ds dt$$

(a) [5] Consider $\underline{x} = (x,y)$ far from the origin, i.e. $\epsilon = 1/|\underline{x}|$ is small. Reproduce the multipole expansion derived in class for the solution u, showing the terms of O(1) and O(ϵ).

(b) [5] Consider the specific case of Ω the unit disk and $f \equiv 1$ in Ω. Evaluate the terms you found in part (a) above. Simplify.

[10] **4.** Consider $u(x,y)$ in the domain $\Omega = \{(x,y) : 0 \leq x \leq 1, 0 \leq y \leq 1\}$ (a square).

(a) [8] Consider the functional

$$F[u] = \int_\Omega u^2 dA$$

for u with zero values on the boundary of Ω, subject to the constraint

$$\int_\Omega (u_x^2 + u_y^2) dA = 1.$$

Find the maximum of F and the maximizing function.

(b) [2] Discuss what occurs when a minimum of F above with its constraint is sought.

[10] **5.** Consider the ODE eigenvalue problem below for $u(x)$, $x \in [0,1]$:

$$-u'' = \lambda a(x) u, \qquad u'(0) = 0, \qquad u'(1) = 0$$

with $a(x) > 0$ given.

Continued on page 3

(a) [3] Note that $\lambda = 0$ is an eigenvalue with eigenfunction $u \equiv 1$. Show that all other eigenvalues are positive.

(b) [7] Show that eigenfunctions u_1 and u_2 corresponding to *different* eigenvalues satisfy

$$\int_0^1 a(x)u_1(x)u_2(x)dx = 0$$

[10] **6.** Consider travel in the plane in which the diagram found on the last page of this exam is embedded. Travel speed $V(x, y)$ is very fast in region A (take $V \to \infty$), slow in region B (take $V \equiv 0$ here) and is scaled to a unit speed outside of A and B. Detach the last page of the exam and sketch the following on it:

(a) [3] The shortest-time travel path (or paths) from point 1 to point 2.

(b) [7] Some contours of $T(x, y)$, the shortest travel time from point 1 to (x,y).

[10] **7.** Consider the ODE BVP for $u(x)$, $x \in [0, 1]$:

$$((1 + x^2)u')' = f(x), \qquad u(0) = 0, \qquad u(1) = 0.$$

(a) [3] Write a weak formulation of this problem.

(b) [7] Let $\{B_i(x)\}$, $i = 1, \dots N-1$ be the piecewise linear finite element basis functions we described in class, on an equally spaced grid with spacing $h = 1/N$ and points $x_i = ih$. Consider

$$U(x) = \sum_{i=1}^{N-1} U_i B_i(x)$$

Write the coefficients of the linear system $\mathbf{A}\underline{U} = \underline{F}$ for the FE approximation of the problem. You may leave the entries of $\mathbf{A}$ and $\underline{F}$ as integrals.

[10] **8.** Consider the ODE eigenvalue problem

$$-u'' + \epsilon x u' = \lambda u, \qquad u(0) = 0, \qquad u(1) = 0.$$

At $\epsilon = 0$, $u = \sin \pi x$ is an eigenfunction corresponding to $\lambda = \pi^2$. Find the O(ϵ) corrections to this u and this λ.

[10] **9.** Consider the following problem for $u(x, y)$ in the infinite plate $x \in \mathbf{R}$, $0 \le y \le 1$:

$$\epsilon u_{xx} + u_{yy} = 0, \qquad u(x, 0) = 0, \qquad u(x, 1) = T(x)$$

where $T \in C^\infty$ is given.

(a) [8] Find the O(1) and O(ϵ) terms in a perturbation series for u.

(b) [2] For what class of functions for T would your solution to (a) fail?

[10] **10.** Consider $\underline{v}(s, t)$ where for each s and t, $\underline{v} \in \mathbf{R}^3$ (a vector with three components). Initial values $\underline{v}(s, 0)$ are given and $\underline{v}$ satisfies

$$\underline{v}_t = \underline{v} \times \underline{v}_{ss}$$

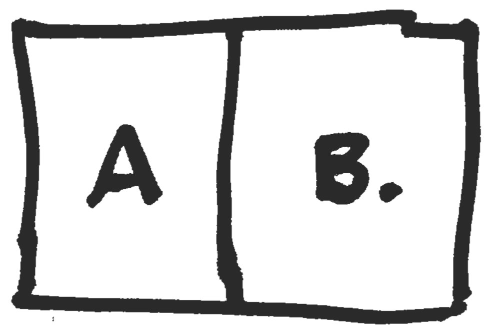
where $\times$ denotes the cross product.

(a) [5] Show that this *nonlinear* problem is well-posed. *Hint:* take the dot product of the equation with $\underline{v}$.

(b) [5] *Either* find the analytic solution of the problem when $\underline{v}(s, 0) = (\cos s, \sin s, 0)$ *or* describe how you would approximate the solution numerically.

The End

1.



2.