MATH 400 – Final exam

Closed book exam; no calculators. Answer as much as you can; credit awarded for the best three answers. Adequately explain the steps you take. *e.g.* if you use an expansion formula, say in one sentence why this is possible; if you quote a special function solution to an ODE, say why this is the correct one. Be as explicit as possible in giving your solutions.

1. Using separation of variables, solve the wave equation,

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right)=u_{tt},$$

inside the unit sphere, $r\le 1$, with the boundary condition,

$$u=0 \quad \text{on} \quad r=1,$$

and initial condition,

$$u(r,\theta,0)=0 \qquad u_t(r,\theta,0)=\cos^3\theta\, g(r).$$

Hint: for the radial part of the problem, the substitution $R(r)=X(r)/\sqrt{r}$, may prove useful, if one sets $u(r,\theta,t)=R(r)Y(\theta)T(t)$.

2. Establish that

$$f\circ g=\mathcal{F}^{-1}\{\hat f\hat g\}, \qquad \mathcal{F}^{-1}\{\hat f(ak)\}=\frac{1}{a}f(x/a) \qquad \text{and} \qquad \mathcal{F}^{-1}\{e^{-|k|}\}=\frac{1}{\pi(1+x^2)}$$

where $\mathcal{F}\{f\}=\hat f(k)$, $\mathcal{F}\{g\}=\hat g(k)$, $f\circ g$ is a convolution, and $a>0$.

Using the Fourier transform, solve the PDE,

$$4u_{xx}+u_{yy}=0, \qquad -\infty<x<\infty, \qquad 0\le y<\infty, \qquad u(x,0)=g(x), \qquad u\to 0 \text{ as } y,|x|\to\infty,$$

expressing your solution in terms of a single integral. Give the result explicitly if $g=1$ for $|x|\le 1$ and $g=0$ for $|x|>1$.

3. Establish the relations,

$$\mathcal{L}\{f(t-a)H(t-a)\}=e^{-as}\overline{f}(s) \qquad \text{and} \qquad \mathcal{L}\{e^{-bt}\}=(b+s)^{-1},$$

for the Laplace transform, where $\overline{f}(s)=\mathcal{L}\{f(t)\}$.

An age-structured model of a population is based on the PDE,

$$u_t+u_a=-\mu(a)u, \qquad 0\le a,t<\infty,$$

where $u(a,t)$ dictates the number of individuals with age a at time t; the death rate $\mu(a)$ is a prescribed function, and initially, $u(a,0)=0$. For age $a=0$, the birth function is

$$u(0,t)=b(t)+\int_0^\infty B(a)u(a,t)da,$$

where $b(t)$ is a prescribed creation function, and $B(a)$ is a prescribed reproductivity.

Using the Laplace transform in time, show that

$$u(a,t) = S(a)\mathcal{L}^{-1}\left\{\frac{\overline{b}(s)e^{-sa}}{D(s)}\right\}, \qquad D(s) = 1 - \int_0^\infty B(a)S(a)e^{-sa}da,$$

where the "survival function,"

$$S(a) = \exp\left[-\int_0^a \mu(a')da'\right].$$

Find an explicit solution for a population for which $B(a) = e^{-a}$, $\mu = 0$, $b(t) = e^{-\nu t}$ and ν is a constant.

4. For

$$u_t - uu_x = 0, \qquad u(x,0) = \cos x.$$

show that an infinite number of shocks form after sufficient time; determine that time and find the shock positions. Draw the characteristic curves on a space-time diagram. Sketch the solution for u upto and beyond the formation of the shock, indicating how can avoid a multivalued solution using an "equal-areas rule". Briefly justify that rule using the integral form of the conservation law corresponding to the PDE and derive a formula for the speed of a shock. For the given initial condition, do the shocks move left, right or stay where they are?

Helpful information:

The Sturm-Liouville differential equation:

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + q(x)y + \lambda\sigma(x)y = 0.$$

Legendre's equation is

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0.$$

Bessel's equation is

$$z^2y'' + zy' + (z^2 - m^2)y = 0,$$

and has the solution, $y = J_m(z)$, which is regular at $z = 0$.

Fourier Transforms:

$$\hat{f}(k) = \mathcal{F}\{f(x)\} = \int_{-\infty}^{\infty} f(x)e^{-ikx}dx, \qquad f(x) = \mathcal{F}^{-1}\{\hat{f}(k)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(k)e^{ikx}dk$$

Laplace Transform:

$$\overline{f}(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}dt.$$

Convolution:

$$f \circ g = \int_{-\infty}^{\infty} f(x')g(x-x')dx'.$$

Helpful trigonometric relations:

$$\cos(A+B) = \cos A\cos B - \sin A\sin B, \qquad \sin(A+B) = \sin A\cos B + \cos A\sin B.$$

MATH 400 – Solution

Side notes: *This was a relatively straighforward exam, with nearly identical questions provided to the students as practice problems during the last week of classes. In view of this, extra effort was required to gain full credit, based on the introductory remarks on page 1 ("Adequately explain the steps you take. e.g. if you use an expansion formula, say in one sentence why this is possible; if you quote a special function solution to an ODE, say why this is the correct one"; see also the specific comments in the solutions). Even with these additional reqirements, about one third of the students did well and received an A.*

1. (11 points) Let $u = R(r)Y(x)T(t)$ where $x = \cos\theta$. Then, separating variables,

$$T'' + \omega^2 T = 0, \qquad [(1-x^2)Y']' + \lambda Y = 0, \qquad (r^2R')' - \lambda R + \omega^2 r^2 R = 0.$$

Thus, T is given by $\sin\omega t$ and Y by $P_n(x)$, with $\lambda = n(n+1)$ and $n = 0,\ 1,\ 2,\ ...$, in view of $u(r,\theta,0) = 0$ and demanding regularity at $x = \pm 1$. Introducing $R = X/\sqrt{r}$ as suggested reduces the $R-$equation to Bessel's equation with $m^2 = \frac{1}{4} + n(n+1) \equiv (n+\frac{1}{2})^2$ and $X = J_m(\omega r)$, after demanding regularity at $r = 0$ which eliminates $Y_m(r)$. But $u(1,\theta,t) = 0$, and so ω must be a zero of $J_m(z)$. Denoting the j^{th} zero of $J_m(z)$ by z_{mj}, we therefore find a general solution,

$$u = r^{-1/2}\sum_{n=0}^{\infty} c_{nj}\sin(z_{mj}t)P_n(x)J_m(z_{mj}r)$$

(7 points so far, including comments as to why one chooses $P_n(x)$ and $J_m(r)/\sqrt{r}$). The coefficients c_{nj} must be chosed to fit the initial condition:

$$u_t(r,\theta,0) = x^3 g(r) \equiv \frac{1}{3}[2P_3(x) + 3P_1(x)]g(r),$$

given that $P_1(x) = x$, $P_n(1) = 1$ and $P_3(x)$ is an odd cubic polynomial (so a little algebra with Legendre's equation gives $P_3 = (5x^3 - 3x)/2$) *(2 points, some indication needed for where the polynomials come from).* Hence, given the Sturm-Liouville form of Bessel's equation,

$$c_{nj} = \frac{\int_0^1 g(r)J_m(z_{mj}r)r^{3/2}dr}{z_{mj}\int_0^1 [J_m(z_{mj}r)]^2 r dr} \times \begin{cases} 3/5 & n = 1 \\ 2/5 & n = 3 \end{cases},$$

and $c_{nj} = 0$ otherwise *(2 points, some indication needed for where the weight functions come from).*

2. (9 points) From the definition of the Fourier transform,

$$\mathcal{F}\{f \circ g\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-ikx}g(x-x')f(x')dxdx' = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-ikz-ikx'}g(z)f(x')dxdz$$

$$\mathcal{F}^{-1}\{\hat{f}(ak)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\hat{f}(ak)dk = \frac{1}{2\pi a}\int_{-\infty}^{\infty} e^{i\kappa(x/a)}\hat{f}(\kappa)d\kappa$$

and

$$\mathcal{F}^{-1}\{e^{-|k|}\} = \frac{1}{2\pi}\int_{-\infty}^{0} e^{ikx+k}dk + \frac{1}{2\pi}\int_{0}^{\infty} e^{ikx-k}dk = \frac{1}{2\pi(ix+1)} - \frac{1}{2\pi(ix-1)},$$

which establish the desired results *(3 points).*

Transforming the PDE:

$$\hat{u}_{yy} = 4k^2\hat{u} \qquad \longrightarrow \qquad \hat{u}(k,y) = \hat{u}(k,0)e^{-2|k|y} = \hat{g}(k)e^{-2|k|y}.$$

Using the results above with $a \equiv y$ and $\hat{f} = e^{-|k|}$ gives

$$u = \frac{2y}{\pi}\int_{-\infty}^{\infty} \frac{g(x')dx'}{4y^2 + (x-x')^2}$$

$$= \frac{1}{\pi}\tan^{-1}\left(\frac{1-x}{2y}\right) + \frac{1}{\pi}\tan^{-1}\left(\frac{1+x}{2y}\right)$$

for the specific example of $g(x)$ *(6 points)*.

3. (10 points) From the definitions,

$$\mathcal{L}\{f(t-a)H(t-a)\} = \int_a^\infty f(t-a)e^{-st}dt = e^{-sa}\int_0^\infty f(\tau)e^{-s\tau}d\tau = e^{-sa}\bar{f}(s)$$

$$\mathcal{L}\{e^{-bt}\} = \int_a^\infty e^{-(s+b)t}dt = \frac{1}{b+s}$$

(2 points).

Given $\mathcal{L}\{u_t(a,t)\} = s\bar{(u)}(a,s) - u(a,0)$ and $u(a,0) = 0$, Laplace transforming the PDE gives

$$\bar{u}_a = -(s+\mu)\bar{u} \qquad \longrightarrow \qquad \bar{u}(a,s) = \bar{u}(0,s)e^{-sa}S(a).$$

(2 points, including explicit incorporation of the initial condition and care with the limits of the integrals).

Taking the transform of the condition at $a = 0$:

$$\bar{u}(0,s) = \bar{b}(s) + \int_0^\infty B(a)\bar{u}(a,s)da \qquad \longrightarrow \qquad \bar{u}(0,s) = \frac{\bar{b}(s)}{D(s)} \quad \& \quad \bar{u}(a,s) = S(a)\frac{\bar{b}(s)e^{-sa}}{D(s)},$$

which gives the desired result. *(3 points)*.

For the sample functions,

$$S(a) = 1, \qquad D(s) = \frac{s}{s+1}, \qquad \bar{b}(s) = \frac{1}{\nu+s},$$

and so

$$\bar{u}(a,s) = \frac{(s+1)}{s(s+\nu)}e^{-sa} = \left[\frac{1}{s} - \frac{1-\nu}{s+\nu}\right]\frac{e^{-sa}}{\nu} \qquad \& \qquad u(a,t) = \frac{1}{\nu}H(t-a)[1-(1-\nu)e^{-\nu(t-a)}].$$

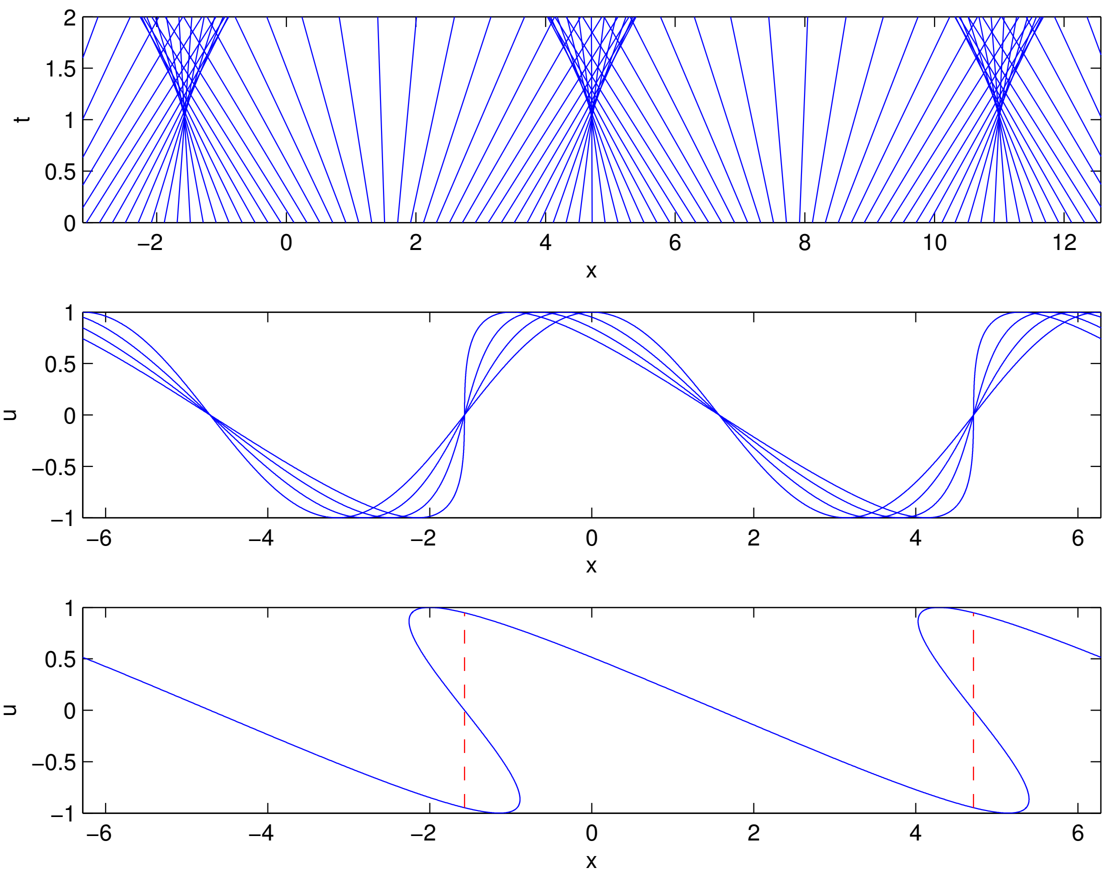
if $\nu \neq 0$, and $u(a,t) = (1+t-a)H(t-a)$ if $\nu = 0$. *(3 points, including explicit consideration of the case $\nu = 0$)*.

4. (11 points) The characteristics equations and solution:

$$\frac{dx}{dt} = -u \quad \& \quad \frac{du}{dt} = 0 \qquad \longrightarrow \qquad x = x_0 - ut \quad \& \quad u = \cos x_0 = \cos(x+ut).$$

Hence,

$$u_x = -\frac{\sin x_0}{1 + t\sin x_0},$$



which first diverges for $x_0 = x = -\frac{1}{2}\pi = \frac{3}{2}\pi = \frac{7}{2}\pi = ...$ at $t = 1$. *(3 points).*

The integral form of the conservation law is

$$\frac{d}{dt}\int_a^b u(x,t)dx = \frac{1}{2}[u^2(x,t)]_a^b$$

For the equal-areas rule, one surgically removes the multivalued part of the solution by introducing a vertical line that cuts out equal area to either side; this is justified by the integral form of the conservation law which demands that $\int_{-\infty}^{\infty} u dx$ (the signed area underneath the curve of u) is constant in time if there is no flux into or out of the full spatial domain ($a \to -\infty$, $b \to \infty$). If u jumps from u^- to u^+ at $x = X(t)$, then

$$\frac{d}{dt}\int_a^X u(x,t)dx + \frac{d}{dt}\int_X^b u(x,t)dx = \int_a^X u_t(x,t)dx + \int_X^b u_t(x,t)dx + (u^+ - u_-)\frac{dX}{dt} = \frac{1}{2}[u^2(x,t)]_a^b$$

Taking the limits $a \to X^-$ and $b \to X^+$ now gives

$$\frac{dX}{dt} = -\frac{1}{2}(u^+ + u^-).$$

For the case at hand, the symmetry of the solution using the method of characteristics implies that $u^- = -u^+$ and so the shocks are stationary, as also implied by the graphical solution. *(5 points).*

Sketches: 3 points.