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The University of British Columbia

MATH 400, Sections 101

Final Exam –December 16, 2016

Name \_\_\_\_\_ Signature \_\_\_\_\_

Student Number \_\_\_\_\_

This exam consists of 6 questions. No notes. Write your answer in the blank page provided.

Problem	max score	score
1.	20	
2.	10	
3.	15	
4.	20	
5.	15	
6.	20	
total	100	

**1. Each candidate should be prepared to produce his library/AMS card upon request.**

**2. Read and observe the following rules:**

No candidate shall be permitted to enter the examination room after the expiration of one half hour, or to leave during the first half hour of the examination.

Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.

CAUTION - Candidates guilty of any of the following or similar practices shall be immediately dismissed from the examination and shall be liable to disciplinary action.

(a) Making use of any books, papers or memoranda, other than those authorized by the examiners.

(b) Speaking or communicating with other candidates.

(c) Purposely exposing written papers to the view of other candidates. The plea of accident or forgetfulness shall not be received.

**3. Smoking is not permitted during examinations.**

(20 points) 1. Solve the following first order fully nonlinear PDE:

$$u_y - yu_x^2 = 0$$

$$u(x, 0) = x, -\infty < x < +\infty$$

You are allowed to use the Charpit's equations

$$\begin{cases} \frac{dx}{ds} = F_p, x(0) = x_0(\xi) \\ \frac{dy}{ds} = F_q, y(0) = y_0(\xi) \\ \frac{dp}{ds} = -F_x - pF_u, p(0) = p_0(\xi) \\ \frac{dq}{ds} = -F_y - qF_u, q(0) = q_0(\xi) \\ \frac{du}{ds} = pF_p + qF_q, u(0) = u_0(\xi) \end{cases}$$

where  $(p_0, q_0)$  is computed via

$$\begin{cases} F(x_0, y_0, p_0, q_0) = 0 \\ u'_0 = p_0 x'_0 + q_0 y'_0 \end{cases}$$

(10 points) 2. Find the general solutions to the following equation

$$2u_{tt} + u_{tx} = x$$

(15 points) 3. Solve the following diffusion equation

$$u_t = u_{xx} + e^t \sin x, 0 < x < \pi, t > 0$$

$$u(x, 0) = 0, 0 < x < \pi$$

$$u(0, t) = 1, u(\pi, t) = 0$$

Hint: you are allowed to use the following formula:

$$u(x, t) = \sum_{n=1}^{\infty} u_n(t) \sin\left(\frac{n\pi}{l}x\right)$$

$$u_n' + k\lambda_n u_n = \frac{2n\pi}{l^2}(h(t) - (-1)^n j(t)) + f_n(t)$$

$$u_n(0) = \phi_n$$

where  $u(0, t) = h(t)$ ,  $u(l, t) = j(t)$ .

(20 points) 4. Consider the following wave equation

$$\begin{aligned}u_{tt} &= u_{xx} \quad 0 < x < 1, \quad t > 0 \\u(x, 0) &= \phi(x), \quad u_t(x, 0) = 0, \quad 0 < x < 1 \\u_x(0, t) + 3u(0, t) &= 0, \quad u_x(1, t) = 0, \quad t > 0\end{aligned}$$

- (i) (15) Use the method of separation of variables to find the general solution. (You should give the equations for negative, zero or positive eigenvalues, and general solution with coefficients in terms of the eigenfunctions and  $\phi$ .)
- (ii) (5) What can you say about the asymptotic behavior of  $u$  as  $t \rightarrow +\infty$ ? Justify your answer.

- (15 points) 5. Use the method of separation of variables to find  $u$  of the following Laplace equation in a sector between two annulus

$$u_{xx} + u_{yy} = 0 \quad \text{in } \{1 < x^2 + y^2 < 4, x > 0, y > 0\}$$

$$u(x, y) = 2x^2 \quad \text{for } x^2 + y^2 = 1, x > 0, y > 0$$

$$u(x, y) = 0 \quad \text{for } x^2 + y^2 = 4, x > 0, y > 0$$

$$u_y(x, 0) = u_x(0, y) = 0, \quad \text{for } 1 < x^2 + y^2 < 4, x \geq 0, y \geq 0$$

Hint: use polar coordinate and the fact that  $\cos^2 \theta = \frac{1 + \cos(2\theta)}{2}$ .

(20 points) 6. Solve the following wave equation in terms of the Bessel function of order 0

$$\begin{cases} u_{tt} = u_{rr} + \frac{1}{r}u_r - 4, & 0 \leq r < 1, t > 0, \\ u_r(1, t) + u(1, t) = 4, & u(r, t) \text{ is bounded,} \\ u(r, 0) = \phi(r), u_t(r, 0) = \psi(r), & 0 \leq r < 1. \end{cases}$$

Hint: find the steady-state solution first.

Here the Bessel function of order 0, denoted by  $J_0(z)$ , is given as the solution to

$$J_0'' + \frac{1}{z}J_0' + J_0 = 0, z \geq 0, \quad J_0(0) = 1$$