Be sure this exam has 5 pages.

THE UNIVERSITY OF BRITISH COLUMBIA

Sessional Examination - December 6, 2011

MATH 340: Linear Programming

Instructor: Dr. R. Anstee, Section 101

Special Instructions: No calculators. You must show your work and **explain** your answers. Quote names of theorems used as appropriate. Time: 3 hours Total marks: 100

1. [12 marks]

a) [10pts] Solve the following linear programming problem, using our standard two phase method and using Anstee's rule.

$$\begin{array}{lrrrl} \text{Maximize} & x_1 & -x_2 & +x_3 & \\ & 2x_1 & -x_2 & & \le -3 \\ & x_1 & -x_2 & +x_3 & \le -2 \\ & & -2x_2 & -x_3 & \le -5 \end{array} \qquad x_1, x_2, x_3 \ge 0$$

b) [2 marks] Give two optimal solutions.

2. [12 marks] Consider the following linear program:

$$\begin{array}{lrrrl} \text{Maximize} & 4x_1 & +5x_2 & +5x_3 & \\ & 2x_1 & +3x_2 & +x_3 & \le 9 \\ & 2x_1 & -x_2 & +3x_3 & \le 7 \\ & 2x_1 & -x_2 & +2x_3 & \le 7 \end{array} \qquad x_1, x_2, x_3 \ge 0$$

a) [2 marks] Give the Dual Linear Program of the above Linear Program.

b) [2 marks] State the Theorem of Complementary Slackness including a description of the conditions of complementary slackness.

c) [6 marks] You are given that an optimal primal solution has $x_1^* = 0$, $x_2^* = 2$, $x_3^* = 3$. Determine an optimal dual solution (without pivoting), stating which theorems you have used.

d) [2 marks] Consider replacing the first inequality of the primal by $2x_1 + 3x_2 + x_3 \le 10$. Does the primal solution $x_1^* = 0$, $x_2^* = 2$, $x_3^* = 3$ and the optimal dual solution determined in c) remain optimal to their new LP's? Explain.

3. [8 marks] Given $A, \mathbf{b}, \mathbf{c}$, current basis (and B^{-1} for your computational ease), use our Revised Simplex method and Anstee's rule to determine the next entering variable (if there is one), the next leaving variable (if there is one), and the new basic feasible solution after the pivot (if there is both an entering and leaving variable). The current basis is $\{x_5, x_2, x_4\}$.

$$\begin{array}{c} \\ x_5 \\ x_6 \\ x_7 \end{array} \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\ 3 & 2 & 1 & 6 & 1 & 0 & 0 \\ 1 & 1 & 2 & 2 & 0 & 1 & 0 \\ -2 & -1 & 1 & -1 & 0 & 0 & 1 \end{pmatrix} \begin{array}{c} \\ x_5 \\ x_6 \\ x_7 \end{array} \begin{pmatrix} \mathbf{b} \\ 4 \\ 1 \\ -1 \end{pmatrix} \qquad B^{-1} = \begin{array}{c} \\ x_5 \\ x_2 \\ x_4 \end{array} \begin{pmatrix} x_5 & x_6 & x_7 \\ 1 & -4 & -2 \\ 0 & -1 & -2 \\ 0 & 1 & 1 \end{pmatrix}$$

$$\mathbf{c}^T \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\ 5 & 1 & 2 & 1 & 0 & 0 & 0 \end{pmatrix}$$

4. [25 marks] An electronic greeting card company wishes to maximize profit by considering what products to make. They can make three possible products made from the three available resources (hours of available staff time) according to the following table. We seek an optimal product mix. We are not concerned with integrality in this question and allow fractional products (we can think of our answer as an optimal product mix not a specific set of products)

	product 1	product 2	product 3	total hours available
artist hours per unit	1	2	1	2
tech hours per unit	2	3	1	3
sales hours per unit	2	1	3	7
$ profit per unit	4	6	3	

Let x_i denote the amount of product i to produce and let x_{3+i} denote the ith slack for $i = 1, 2, 3$. The final dictionary is:

$$\begin{array}{rcrrrr} x_1 & = & 1 & -x_2 & +x_4 & -x_5 \\ x_3 & = & 1 & -x_2 & -2x_4 & +x_5 \\ x_6 & = & 2 & +4x_2 & +4x_4 & -x_5 \\ z & = & 7 & -x_2 & -2x_4 & -x_5 \end{array} \qquad B^{-1} = \begin{array}{c} \\ x_1 \\ x_3 \\ x_6 \end{array} \begin{array}{c} \begin{array}{ccc} x_4 & x_5 & x_6 \end{array} \\ \begin{pmatrix} -1 & 1 & 0 \\ 2 & -1 & 0 \\ -4 & 1 & 1 \end{pmatrix} \end{array}$$

NOTE: All questions are independent of one another.

a) [3 marks] Give the marginal values for each of the resources: artist hours, tech hours and sales hours.

b) [2 marks] Give the range on c_2 (profit for product 2) so that the current solution remains optimal.

c) [5 marks] Give the range on c_3 (profit for product 3) so that the current solution remains optimal.

d) [5 marks] Find the range on p so that we may change the availability of each labour resource by p units (i.e. change to $2 + p$ artist hours, $3 + p$ tech hours and $7 + p$ sales hours) and still have the same optimal basis $\{x_1, x_3, x_6\}$. Also give the profit as a linear function of p in that range.

Hint for e),f): You need not complete all of the very final dictionary, merely the variables in the basis and the constants and **all** the entries in the z row.

e) [5 marks] Change the resource availabilities to 3 for artist hours, 4 for tech hours and 6 for sales hours. Determine the new optimal solution using the Dual Simplex method. Report the new solution as well as the new marginal values.

f) [5 marks] Consider adding a new constraint $2x_1 + x_2 + x_3 \leq 2$ to our original problem. Solve using the Dual Simplex method. Report the new solution as well as the new marginal values.

5. [12 marks] We have machines that provide 3 services, each in varying amounts. Our table gives the amount of service i from machine j when running machine j for one hour. We wish to meet the customer demand for services as given. Let x_i denote the number of hours we are using machine i, for $i = 1, 2, 3$. We are not concerned with integrality in this question.

	one hour machine 1	one hour machine 2	one hour machine 3	customer demand
service 1	5	6	8	50
service 2	6	9	7	63
service 3	8	10	15	72
profit/hour	20	25	38	

Our profit coefficients for using machine i depends on all services from a machine i being used to meet demand. We would like the supply of each service to exactly match the customer demand but this seems difficult (you can argue this is impossible for this data). We introduce the possibility of reducing demand for each service. Let d_i denote the reduction in demand for service i where, for example if $d_3 = 1$, we are reducing the demand for service 3 by 1 unit. We are required to pay a 'penalty' of 5 for each unit of demand reduction d_i. This is a penalty for not meeting customer demand.

We model the problem using an LP. The following is the input file for LINDO:

```
max 20x1+25x2+38x3-5d1-5d2-5d3
subject to
service1)5x1+6x2+8x3+d1=50
service2)6x1+9x2+7x3+d2=63
service3)8x1+10x2+15x3+d3=72
end
```

The useful LINDO output is on the next page.

a) [3 marks] For which individual services, if any, would you like to see an increase in customer demand. Explain.

b) [3 marks] What is the effect on profit if the customer demand for service 1 increases by 7 units?

c) [3 marks] Consider the possibility of raising the demand of service 3 by e_3 units at a cost of 4 per unit. How would you formulate the LP to maximum the profit subject to this additional flexibility. Do not solve.

d) [3 marks] Explain why 5 plus the dual price of the second constraint (SERVICE2) is equal to the reduced cost of d_2.

```
  LP OPTIMUM FOUND AT STEP      0

       OBJECTIVE FUNCTION VALUE

       1)      147.5000

 VARIABLE        VALUE          REDUCED COST
       X1         1.500000          0.000000
       X2         6.000000          0.000000
       X3         0.000000          9.916667
       D1         6.500000          0.000000
       D2         0.000000          4.166667
       D3         0.000000         11.250000


      ROW   SLACK OR SURPLUS     DUAL PRICES
SERVICE1)         0.000000         -5.000000
SERVICE2)         0.000000         -0.833333
SERVICE3)         0.000000          6.250000

NO. ITERATIONS=       0


RANGES IN WHICH THE BASIS IS UNCHANGED:

                          OBJ COEFFICIENT RANGES
VARIABLE         CURRENT        ALLOWABLE        ALLOWABLE
                   COEF          INCREASE         DECREASE
      X1       20.000000         5.000000         1.830769
      X2       25.000000         3.500000         6.250000
      X3       38.000000         9.916667         INFINITY
      D1       -5.000000         4.760000        24.999998
      D2       -5.000000         4.166667         INFINITY
      D3       -5.000000        11.250000         INFINITY

                          RIGHTHAND SIDE RANGES
     ROW         CURRENT        ALLOWABLE        ALLOWABLE
                   RHS           INCREASE         DECREASE
SERVICE1       50.000000         INFINITY         6.500000
SERVICE2       63.000000         1.800000         9.000000
SERVICE3       72.000000         8.666667         2.000000
```

6. [12 marks]

a) [2 marks] Let $\mathbf{1}$ denote the $n \times 1$ vector of 1's. Let $\mathbf{x} \geq \mathbf{0}$ be an $n \times 1$ vector. Show that $\mathbf{1} \cdot \mathbf{x} > 0$ if and only if $\mathbf{x} \neq \mathbf{0}$.

b) [10 marks] Let A be a given $m \times n$ matrix, let $\mathbf{b}$ be a $m \times 1$ vector. Show that either

i) There exist an $\mathbf{x} \neq \mathbf{0}$ with $A\mathbf{x} = \mathbf{0}$, $\mathbf{x} \geq \mathbf{0}$,

or

ii) There exists a $\mathbf{y}$ with $A^T\mathbf{y} \geq \mathbf{1}$,

but not both.

You may use the result from a) even if you did not prove it.

7. [4 marks] Let A be a matrix satisfying $A > 0$ which is the same as saying each entry of A is strictly greater than 0. Let $\mathbf{b} \geq \mathbf{0}$ and $\mathbf{c} \geq \mathbf{0}$ be some given vectors. Then show that the LP

$$\begin{array}{rc} \max & \mathbf{c} \cdot \mathbf{x} \\ & A\mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{array} ,$$

always has an optimal solution.

8. [10 marks] Let $A = (a_{ij})$ be an $m \times n$ matrix representing the payoff to the row player where a_{ij} is the payoff to the row player if the row player plays strategy i and the column player plays strategy j. A mixed strategy for the row player can be given as $\mathbf{x} = (x_1, x_2, \ldots, x_m)^T$.

a) [5 marks] Assume that for some given k, ℓ that the kth row of A is at most the ℓth row of A, namely $a_{kj} \leq a_{\ell j}$ for $j = 1, 2, \ldots, n$. Show that there is an optimal strategy $\mathbf{x}$ for the row player with $x_k = 0$.

b) [5 marks] Assume for some given k, ℓ that the kth row of A is strictly less than the ℓth row of A, namely $a_{kj} < a_{\ell j}$ for $j = 1, 2, \ldots, n$. Show that any optimal strategy $\mathbf{x}$ for the row player has $x_k = 0$.

9. [5 marks] Consider an LP in the following standard form:

$$\begin{array}{rc} \max & \mathbf{c} \cdot \mathbf{x} \\ & A\mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{array} .$$

Let b_i denote the ith entry of $\mathbf{b}$. We have found the optimal solution and optimal basis B using LINDO. We have discovered by considering the LINDO output that the same basis is optimal if we increase b_1 by 4 and also the same basis is optimal if we increase b_2 by 6. Show that the same basis is optimal if we increase b_1 by 2 and simultaneously increase b_2 by 3. You might note that

$$\frac{1}{2}\begin{bmatrix} 4 \\ 0 \\ 0 \\ \vdots \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 \\ 6 \\ 0 \\ \vdots \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 0 \\ \vdots \end{bmatrix} .$$