Be sure that this examination has 12 pages including this cover

The University of British Columbia

Final Examinations - April 2007

Mathematics 340–202

Closed book examination   Time: 2.5 hours

Name _____   **Signature** _____

Student Number_____   **Instructor's Name** _____

                                                   Section Number _____

Special Instructions:

Calculators, notes, or other aids may not be used. Answer questions on the exam. Note that the exam is two-sided! A note sheet is provided with the exam. You have two extra pages at the back for additional space.

1		8
2		8
3		20
4		16
5		12
6		28
Total		92

Marks

[8]   **1.**   Consider a zero sum game with M being player A's payout matrix.

(a) Show that for any matrix M and probability vectors $\vec{\alpha}, \vec{\beta}$ (of appropriate dimension) we have
$$\mathrm{Scream}_A(\vec{\alpha}) \leq \vec{\alpha}^{\mathrm{T}} M \vec{\beta} \leq \mathrm{Scream}_B(\vec{\beta}) \tag{1}.$$

(b) Assume equation (1) always holds. Show that if
$$\mathrm{Scream}_A(\vec{\alpha}) = \mathrm{Scream}_B(\vec{\beta}) \qquad \text{and} \qquad \mathrm{Scream}_A(\vec{\alpha}') = \mathrm{Scream}_B(\vec{\beta}')$$

then
$$\mathrm{Scream}_A(\vec{\alpha}) = \mathrm{Scream}_A(\vec{\alpha}')$$

[8]   **2.**   Use the two-phase method to solve:

$$\text{maximize} \quad x_1 + x_2, \quad \text{subject to } x_1, x_2 \geq 0 \text{ and}$$
$$x_1 + 2x_2 \leq 10$$
$$-x_1 - 2x_2 \leq -4$$

When there is more than one possible entering or leaving variable, choose the one with the smallest subscript.

[20]   **3.**   Consider the problem maximize $x_1 + 2x_2$ subject to $x_1 + x_2 \leq 1$ and $x_1, x_2 \geq 0$.

(a) Write the dual LP.

(b) Consider a proposed optimal solution $x_1 = 1$ and $x_2 = 0$. Use complementary slackness to find the dual solution to this proposed solution. Does the dual solution verify that the propsed solution is optimal?

(c) Same questions as in part (b) for a proposed optimal solution $x_1 = 0$ and $x_2 = 1$.

(d) Consider the problem maximize $x_1 + 2x_2$ subject to $x_1 + x_2 \leq 1$ and $x_1 + x_2 \leq 1$ and $x_1, x_2 \geq 0$ (yes, this is essentially the same LP, except that the constraint $x_1 + x_2 \leq 1$ appears twice). Write the dual LP. Will complementary slackness always yield a unique dual solution? Explain.

(e) Consider the LP maximize $x_1 + x_2$ subject to $2x_1 \leq 1$ and $x_1 + x_2 \leq 1$ and $2x_2 \leq 1$ and $x_1, x_2 \geq 0$. Will complementary slackness always yield a unique dual solution? Explain. (You are not required to write down the dual LP if you can argue convincingly without it.)

[16]   **4.**   Let B_i denote the basis variables in the i-th dictionary of the revised simplex method. Say that $A_{B_7} = 2I$ with I the identity matrix, and that

$$A_{B_8} = A_{B_7}E, \qquad A_{B_9} = A_{B_8}F,$$

where

$$E = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{bmatrix}, \qquad F = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

(a)   Compute $c_B^T A_{B_9}^{-1}$ as you would in the revised simplex method, where

$$c_B^T = \begin{bmatrix} 7 & 8 & 9 \end{bmatrix}.$$

(b)   Let A, B, C be matrices that are $1 \times n$, $n \times n$, and $n \times n$ respectively. How many multiplications are required to multiply ABC, if we first multiply AB? How many if we first multiply BC? Which method requires fewer multiplications when n is large? Assume that you multiply matrices naively (taking dot products of each row of the first matrix with the columuns of the second matrix).

(c)   In the revised simplex method, explain why the constants in the $\vec{x}_B$ row, namely $A_B^{-1}\vec{b}$, should be easily available from the previous iteration. [Hint: this also holds for the plain old simplex method.]

[12]   **5.**   Consider the LP maximize $x_1 + ax_2$ subject to $x_1 + x_2 \le 1$ and $x_1 + 2x_2 \le b$ and $x_1, x_2 \ge 0$, where a and b are real parameters. Introducing slack variables, x_3, x_4 (x_3 for the first inequality), here is one possible dictionary:

$$
\begin{aligned}
x_1 &= & 2 - b & & -2x_3 & & +x_4 \\
x_2 &= & b - 1 & & +x_3 & & -x_4 \\
z &= & 2 - a - b + ab & & +(a-2)x_3 & & +(1-a)x_4
\end{aligned}
$$

(a) For what range of a, b is the above dictionary optimal? Explain.

(b) Perform one regular simplex pivot to obtain a dictionary optimal when $b = 3/2$ and $a = 3$. Explain how you are choosing entering and leaving variables. For what range of a, b is this dictionary optimal?

(c) Perform one dual pivot to obtain a dictionary optimal when $b = 3$ and $a = 3/2$. Explain how you are choosing entering and leaving variables. For what range of a, b is this dictionary optimal?

[28] **6.** Consider the matrix game associated to the matrix

$$M = \begin{bmatrix} 1 & 2 \\ 3 & c \end{bmatrix},$$

where c is a given real number.

(a) Assuming that all pure strategies are involved in a unique equilibrium, what is player A's equilibrium strategy, $\vec{\alpha} = (\alpha_1, \alpha_2)$?

(b) For what values of c is it not the case that all pure strategies are involved in an equilibrium? Use domination to explain what are the equilibria for those values of c.

(c) Consider a general LP, maximize $\vec{c}^{\mathrm{T}}\vec{x}$ subject to $A\vec{x} \leq \vec{b}$ and $\vec{x} \geq \vec{0}$. If the third inequality reads $x_1 + x_2 \leq 1$, and the fourth reads $x_1 + x_2 \leq 2$, can you say that one inequality "dominates" another in some reasonable sense? Explain. Same question with $3x_1 + 3x_2 \leq 1$ and $x_1 + 2x_2 \leq 2$ (recall that x_1, x_2 are both non-negative!).

(d) Write an LP for MAXScream$_A$ in the matrix game above. Explain how one inequality "dominates" another (in your sense) for $c \geq 2$.

(e) Consider a general LP with an optimal solution in which a slack variable is positive. Can we discard the corresponding inequality and obtain the same optimal objective? Explain. Similarly, can we discard a decision variable that is zero? Explain.

(f) Consider an LP with an optimal solution in which every slack variable is zero
and every decision variable is positive. Can there be more decision variables than
slack variables? [Hint: in a dictionary, how many variables are basic?]

(g) Explain how part (e) and (f) can reduce any to LP to linear algebra provided
that you can guess which primal and dual decision variables can be 0 in certain
optimal (primal and dual) solutions. Explain what this has to do with a simi-
lar observation made in class (and the notes) about matrix games that motives
part (a) of this problem.

The End