THE UNIVERSITY OF BRITISH COLUMBIA
SESSIONAL EXAMINATIONS – DECEMBER 2008
MATHEMATICS 322

TIME: 2 1/2 hours

1. [16 marks]

a) Explain what is meant by the "centre" $Z(G)$ of a group G.

b) Prove that $Z(G)$ is a subgroup of G.

c) If a subgroup H of a group G is contained in $Z(G)$, show that H is a normal subgroup of G.

d) If ϕ is a surjective homomorphism from a group G to a group K, show that

$$L = \{k \in K \mid k = \phi(g) \text{ for some } g \in Z(G)\}$$

is a subgroup of K and that it is contained in $Z(K)$.

2. [10 marks] For n a fixed positive integer, let G be the group of invertible upper-triangular $n \times n$ matrices with entries in $\mathbb{C}$ under multiplication of matrices (you may assume that G is a group). Prove that the set N of elements of G whose diagonal entries are equal to ± 1 or $\pm i$ (where i is as usual a square root of -1) is a normal subgroup of G and that the quotient group G/N is abelian.

3. [10 marks]

a) Show that the dihedral group D_{20} has a subgroup that is isomorphic to D_{10}.

b) Explain why the subgroup in part a) is a normal subgroup of D_{20}.

4. [15 marks] Determine, with explanation, whether the following statements are true or false:

a) The groups $U(\mathbb{I}_{10})$ and $\mathbb{I}_4$ are isomorphic.

b) If G is a nonabelian group and N is a normal subgroup of G, then the quotient group G/N is also nonabelian.

c) If G is a group of order 24 that has 5 conjugacy classes, of sizes 1, 3, 6, 6, and 8, then G has exactly two normal subgroups.

5. [10 marks] Let R be the polynomial ring $\mathbb{F}_2[x]$.

a) Show that if I is an ideal of R then $J = \{y \in R \mid y^2 \in I\}$ is an ideal of R.

b) If I is the principal ideal of R generated by $x^3 + x^2$, find a generator of J.

6. [18 marks] Factor the following elements of the given rings R into irreducible elements of R. Explain why the factors you give are irreducible. In each case, you may assume that R is a ring.

a) 15, an element of $R = \{a + b\sqrt{-2} \mid a, b \in \mathbb{Z}\}$

b) $5x^4 - 20x^3 + 30$, an element of $R = \mathbb{Z}[x]$

c) $x^3 + 1$, an element of $R = \mathbb{F}_7[x]$

7. [15 marks] Let R be the set of complex numbers of the form $a + ib$, where a and b are integers. You may assume that R is a ring under the usual addition and multiplication of complex numbers.

a) Show that $I = \{a + ib \mid a \text{ is even and } b \text{ is odd}\}$ is not an ideal of R.

b) Show that the set J which consists of all elements $a + ib$ such that a and b are either both even or both odd *is* an ideal of R.

c) Explain why the ideal J above is principal and find a generator of it.

8. [6 marks] Prove that there is no simple group of order 28.