Math 321 Final Exam

Apr 20, 2009

Duration: 150 minutes

Name: ______________________ Student Number: ______________________

Section: ______________________

Do not open this test until instructed to do so! This exam should have 19 pages, including this cover sheet. No textbooks, calculators, or other aids are allowed. Turn off any cell phones, pagers, etc. that could make noise during the exam. You must remain in this room until you have finished the exam. **Circle your solutions! Reduce your answer as much as possible. Explain your work.** Relax. Use the back of the page if necessary.

Read these UBC rules governing examinations:

(i) Each candidate must be prepared to produce, upon request, a Library/AMS card for identification.

(ii) Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.

(iii) No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination.

(iv) Candidates suspected of any of the following, or similar, dishonest practices shall be immediately dismissed from the examination and shall be liable to disciplinary action.

- Having at the place of writing any books, papers or memoranda, calculators, computers, audio or video cassette players or other memory aid devices, other than those authorized by the examiners.
- Speaking or communicating with other candidates.
- Purposely exposing written papers to the view of other candidates. The plea of accident or forgetfulness shall not be received.

(v) Candidates must not destroy or mutilate any examination material; must hand in all examination papers; and must not take any examination material from the examination room without permission of the invigilator.

Problem	**Out of**	**Score**
1	15	
2	12	
3	15	
4	20	
5	12	
6	16	
7	20	
Total	110	

Problem 1 (15 points)

Below $a, b \in \mathbb{R}$ with $a < b$.

(a) Define carefully what it means for a function f to be of bounded variation on $[a, b]$.

(b) When do we say that a function sequence $\{f_n\}_{n\in\mathbb{N}}$ is uniformly bounded on $[a, b]$?

(c) When do we say that a function sequence $\{f_n\}_{n\in\mathbb{N}}$ is equicontinuous on $[a, b]$?

Problem 2 (12 points)

Let α be increasing and $f \in \mathcal{R}(\alpha)$ on $[a, b]$. Denote by m and M the infimum and supremum of $\{|f(x)| : x \in [a, b]\}$ respectively.

(a) Show that there exists $c \in [m, M]$ such that

$$\int_a^b f(x) d\alpha = c[\alpha(b) - \alpha(a)].$$

(b) If, in addition, f is continous on $[a, b]$, prove that there exists $x_0 \in [a, b]$ such that

$$\int_a^b f(x) d\alpha = f(x_0)[\alpha(b) - \alpha(a)].$$

Problem 3 (15 points)

(a) Let α be increasing and assume that $f \in \mathcal{R}(\alpha)$ on $[a, b]$. Define F as

$$F(x) := \int_a^x f d\alpha \quad \text{for } x \in [a, b].$$

Show that F is of bounded variation on $[a, b]$. [Hint: The mean value theorem that you proved in the previous problem may be useful.]

(b) For F defined above, show that F is continuous at every point at which α is continous.

(c) Above, if we replace the assumption that α is increasing with the assumption that α is of bounded variation on $[a, b]$, would the conclusions of parts (b) and (c) still valid? Give a **brief** argument.

Problem 4 (20 points)

Give examples of each of the following together with a brief explanation. Make your examples as simple as possible. Sketch the graphs of the functions involved if feasible.

(a) $f_n \to f$ in the mean (i.e., in L^2), but not pointwise or uniformly.

(b) $f_n \to f$ uniformly, but not in L^2.

(c) $f_n \to f$ uniformly, all of the f_n and f are integrable, but $\int_{-\infty}^{\infty} f_n(x)dx$ **does not converge** to $\int_{-\infty}^{\infty} f(x)dx$.

(d) A bounded function f and an increasing function α defined on $[0, 1]$ such that $|f| \in \mathcal{R}(\alpha)$ but for which $\int_0^1 f d\alpha$ does not exist.

Problem 5 (12 points)

Prove that if $f : [0, 1] \to \mathbb{R}$ is a continuous function that obeys

$$\int_0^1 f(x)\, x^n \; dx = 0$$

for all $n \in \mathbb{Z}$ with $n \geq 0$, then $f(x)$ is identically zero. [Hint: use Weierstrass approximation theorem to prove that $\int_0^1 f(x)^2 \; dx = 0$.]

Problem 6 (16 points) Let f be 2π-periodic function which on $[0, 2\pi)$ satisfies

$$f(t) = \begin{cases} 1 & \text{if } 0 \le t \le \pi, \\ 0 & \text{if } \pi < t < 2\pi. \end{cases}$$

(a) For $k \in \mathbb{Z}$, let $\widehat{f}(k)$ denote the kth Fourier coefficient of f, i.e., $\widehat{f}(k) = \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-ikt}dt$. Show that

$$\widehat{f}(k) = \begin{cases} \frac{1}{i\pi k} & \text{if } k \text{ is an odd integer,} \\ \frac{1}{2} & \text{if } k = 0, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

(b) Use (1) and Parseval's identity to show that

$$\sum_{n=1}^{\infty} \frac{1}{(2k-1)^2} = \frac{\pi^2}{8}.$$

(c) For $a, b, c \in \mathbb{R}$, define

$$G(a, b, c) := \int_0^{2\pi} \left| f(t) - (a + be^{it} + ce^{-it}) \right|^2 dt.$$

where f is as in part (a). Find a_0, b_0, c_0 such that

$$G(a_0, b_0, c_0) \le G(a, b, c), \quad \forall a, b, c \in \mathbb{R}.$$

(Justify your answer, i.e., identify all theorems that you use.)

Problem 7 (20 points)

In this problem, we will prove Fejér's theorem in several steps. If you are stuck with one part, move on to the next (and feel free to use the results from earlier parts).

(a) Let $x \in \mathbb{R}$ that is not an integer multiple of π. Show that

$$\sum_{k=0}^{n-1} e^{i(2k+1)x} = \frac{\sin(nx)}{\sin x} e^{inx}. \tag{2}$$

Using (2), prove that

$$\sum_{k=0}^{n-1} \sin[(2k+1)x] = \frac{\sin^2(nx)}{\sin x}.$$

(b) Let D_n be the Dirichlet kernel given on $[-\pi, \pi]$ by

$$D_n(x) = \begin{cases} \dfrac{\sin[(n+\frac{1}{2})x]}{\sin\frac{x}{2}} & \text{if } x \neq 0 \\ 2n+1 & \text{if } x = 0. \end{cases}$$

Define, now, F_n on $[-\pi, \pi]$ via

$$F_n(x) := \frac{1}{n} \sum_{k=0}^{n-1} D_k(x).$$

Show that

$$F_n(x) = \begin{cases} \dfrac{1}{n} \dfrac{\sin^2(\frac{nx}{2})}{\sin^2(\frac{x}{2})} & \text{if } x \neq 0 \\ n & \text{if } x = 0. \end{cases}$$

(c) Let f be a 2π-periodic function that is Riemann integrable on $[-\pi, \pi]$. Define

$$\sigma_n(f;x) := \frac{1}{n}\big(s_0(f;x) + s_1(f;x) + \cdots + s_{n-1}(f;x)\big)$$

where $s_j(f;x)$ is the jth partial sum of the Fourier series of f. Show that

$$\sigma_n(f;x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t) F_n(t) dt.$$

[Hint: recall that $s_n(f;x) = \frac{1}{2\pi} \int_{\pi}^{\pi} f(x-t) D_n(t) dt$.]

(d) Show that for any positive integer n,

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} F_n(x) dx = 1.$$

[Hint: Apply (c) to the constant function $f = 1$.]

(e) Suppose that f is a 2π-periodic function that is continuous on $[-\pi, \pi]$. Show that $\sigma_n(f;x)$ converges to $f(x)$ pointwise. Moreover, show that the convergence is uniform. [Hint: Define $g_x(t) := f(x-t) - f(x)$ and write $f(x) - \sigma_n(f;x)$ in terms of g_x.]