

**Final Examination — December 19, 2015**      **Duration: 2.5 hours***This test has 8 questions on 9 pages, for a total of 80 points.*

Dr. G. Slade

- Read all the questions carefully before starting to work.
- Continue on the back of the previous page if you run out of space, *with clear indication on the original page* that your solution is continued elsewhere.
- This is a closed-book examination. **No aids of any kind are allowed**, including: documents, cheat sheets, electronic devices of any kind (including calculators, phones, etc.)

First Name: \_\_\_\_\_ Last Name: \_\_\_\_\_

Student-No: \_\_\_\_\_ Section: \_\_\_\_\_

Signature: \_\_\_\_\_

Question:	1	2	3	4	5	6	7	8	Total
Points:	10	10	10	10	10	10	10	10	80
Score:									

**Student Conduct during Examinations**

- Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCcard for identification.
- Examination candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.
- No examination candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no examination candidate shall be permitted to enter the examination room once the examination has begun.
- Examination candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.
- Examination candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
  - speaking or communicating with other examination candidates, unless otherwise authorized;
  - purposely exposing written papers to the view of other examination candidates or imaging devices;
  - purposely viewing the written papers of other examination candidates;
  - using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
  - using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s) (electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).
- Examination candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.
- Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.
- Examination candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

10 marks
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1. Define the following.

(a)  $X$  is a *metric space* with *metric*  $d$ .

(b) The subset  $E$  of a metric space  $X$  is *dense*.

(c) An *open cover* of a subset  $E$  of a metric space  $X$ .

(d) A *connected* subset  $E$  of a metric space  $X$ .

(e) The series  $\sum_{n=1}^{\infty} a_n$  *converges and equals*  $A$  (here each  $a_n$  is a complex number).

2. Let  $A_1, A_2, \dots$  be subsets of a metric space  $X$ .

6 marks

(a) For  $n = 1, 2, \dots$ , let  $B_n = \cup_{i=1}^n A_i$ . Prove that  $\overline{B_n} = \cup_{i=1}^n \overline{A_i}$ .

2 marks

(b) Let  $B = \cup_{i=1}^{\infty} A_i$ . Prove that  $\overline{B} \supset \cup_{i=1}^{\infty} \overline{A_i}$ .

2 marks

(c) Give an example to show that the inclusion in part (b) can be proper.

3. Let  $B \subset \mathbb{R}^k$  be nonempty and compact, and  $a \in B^c$ . The metric on  $\mathbb{R}^k$  is the usual metric  $d(x, y) = |x - y|$ . Recall that the distance  $d(a, B)$  from  $a$  to  $B$  is defined by  $d(a, B) = \inf\{d(a, b) : b \in B\}$ .

8 marks

- (a) Prove that there exists  $b \in B$  such that  $d(a, b) = d(a, B)$ .

2 marks

- (b) Is the  $b$  in part (a) unique? Give a short proof if so, or a counterexample if not.

3 marks
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4. (a) Let  $\{s_n\}$  be a sequence in  $\mathbb{R}$ . Recall the definitions  $\liminf_{n \rightarrow \infty} s_n = \sup_{n \geq 1} \inf_{m \geq n} s_m$  and  $\limsup_{n \rightarrow \infty} s_n = \inf_{n \geq 1} \sup_{m \geq n} s_m$ . Consider the sequence  $a_1, a_2, a_3, \dots$  in  $\mathbb{Z}$  which starts with 1, 6, 2, 12, 4, 24, 8, 48, 16, 96,  $\dots$  and continues this pattern. Determine each of the following six values, or state that they do not exist:

$$L_1 = \liminf_{n \rightarrow \infty} a_{n+1}/a_n, \quad L_2 = \limsup_{n \rightarrow \infty} a_{n+1}/a_n, \quad L_3 = \lim_{n \rightarrow \infty} a_{n+1}/a_n,$$

$$M_1 = \liminf_{n \rightarrow \infty} a_{n+2}/a_n, \quad M_2 = \limsup_{n \rightarrow \infty} a_{n+2}/a_n, \quad M_3 = \lim_{n \rightarrow \infty} a_{n+2}/a_n.$$

3 marks
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- (b) Let  $X$  be a metric space and suppose that the sequence  $\{s_n\}$  in  $X$  converges. Prove that  $\{s_n\}$  is a Cauchy sequence.

4 marks
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- (c) Let  $X$  be the set of all sequences of real numbers for which only finitely many members of the sequence are nonzero; thus an element  $\vec{x} \in X$  is a sequence  $\vec{x} = \{x_i\}$  with each  $x_i \in \mathbb{R}$  and with only finitely many  $x_i \neq 0$ . A metric is defined on  $X$  by  $d(\vec{x}, \vec{y}) = [\sum_{i=1}^{\infty} |x_i - y_i|^2]^{1/2}$  (you need not prove that this is a metric). Consider the sequence  $\{\vec{x}_n\}$  in  $X$  defined by  $\vec{x}_n = (1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, 0, 0, 0, \dots)$ . Prove that  $\{\vec{x}_n\}$  is a Cauchy sequence in  $X$ .

10 marks
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5. Let  $E \subset \mathbb{R}$  be a bounded set, and suppose that  $f : E \rightarrow \mathbb{R}$  is uniformly continuous. Prove that  $f(E)$  is bounded.

5 marks
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6. (a) Does the series  $\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$  converge or diverge?

5 marks
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(b) For which values of  $z \in \mathbb{C}$  does the series  $\sum_{n=0}^{\infty} \frac{1}{1+2n^2} z^n$  converge and for which values of  $z$  does it diverge?

10 marks
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7. Suppose that  $f : [0, \infty) \rightarrow \mathbb{R}$  is continuous, that  $f(0) = 0$ , and that  $f$  is differentiable on  $(0, \infty)$  with  $f'$  monotonically increasing. Let  $h(x) = x^{-1}f(x)$ . Prove that  $h$  is monotonically increasing.



10 marks
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8. Use Taylor's Theorem to approximate  $\sqrt{e}$  to within an error of magnitude at most  $10^{-3}$ .  
(It is sufficient to express your approximation as a sum of rational numbers, it is not necessary to do the arithmetic to simplify fully.)