Math 320, Fall 2007

Final, December 17

Name: SID:

Instructions

- The total time is 2 hours and 30 minutes.
- The total score is 100 points.
- Use the reverse side of each page if you need extra space.
- Show all your work. A correct answer without intermediate steps will receive no credit.
- Calculators and cheat sheets are not allowed.

Problem	Points	Score
1	15	
2	10	
3	20	
4	15	
5	15	
6	25	
TOTAL	100	

1. Recall that a function $f : (M, d) \to (M, d)$ is called a *strict contraction* if there exists a constant $\alpha < 1$ such that

$$d(f(x), f(y)) \leq \alpha d(x, y) \qquad \text{for all } x, y \in M.$$

(a) Is a strict contraction continuous? Uniformly continuous?

(3 points)

(b) Show that for every $x \in M$, the sequence of functional iterates $\{f^{(n)}(x) : n \in \mathbb{N}\}$ is a Cauchy sequence.

(5 points)

(c) Use parts (a) and (b) to show that if M is complete and f is a strict contraction, then f has a fixed point, i.e., there exists $x_0 \in M$ such that $f(x_0) = x_0$.

(5 points)

(d) Is the fixed point in part (c) unique?

(2 points)

2. Recall the normed vector space

$$\ell_p = \left\{ \mathbf{x} = (x_1, x_2, \cdots) : \sum_{n=1}^{\infty} |x_n|^p < \infty \right\},$$

$$\text{with norm } ||\mathbf{x}||_p = \left[\sum_{n=1}^{\infty} |x_n|^p \right]^{\frac{1}{p}}.$$

Is the set

$$A = \left\{ \mathbf{x} \in \ell_3 : |x_n|^3 \leq \frac{1}{n} \text{ for all } n \in \mathbb{N} \right\}$$

compact in ℓ^3?

(10 points)

3. (a) Describe the closure of the set

$$E = \left\{ \left(x, \cos\left(\frac{1}{x^2}\right) \right) : 0 < x \leq 1 \right\} \subseteq \mathbb{R}^2.$$

(5 points)

(b) Is the set $\overline{E}$ path-connected, where E is as in part (a)? Explain.
(10 points)

(c) Repeat the question in part (b) for the closure of the set

$$F = \left\{ \left(x, \arctan\left(\frac{1}{x^2}\right) \right) : 0 < x \leq 1 \right\}.$$

(5 points)

4. (a) Recall the Hilbert cube

$$\mathbb{H}^\infty = \{\mathbf{x} = (x_1, x_2, \cdots) : |x_n| \leq 1 \text{ for all } n \in \mathbb{N}\}$$

$$\text{with metric } d(\mathbf{x}, \mathbf{y}) = \sum_{n=1}^{\infty} 2^{-n}|x_n - y_n|.$$

Show that $\mathbb{H}^\infty$ is separable, i.e., there exists a countable subset $A \subseteq \mathbb{H}^\infty$ such that $\overline{A} = \mathbb{H}^\infty$.

(10 points)

(b) Give an example of a non-separable metric space, with reasons for your answer.

(5 points)

5. Suppose $\{a_n : n \in \mathbb{N}\}$ is a sequence of strictly positive numbers such that $\sum_{n=1}^{\infty} a_n$ diverges. Let s_n denote the n-th partial sum of this series, i.e., $s_n = a_1 + \cdots + a_n$.

(a) Determine the convergence or divergence of $\sum_{n=1}^{\infty} a_n/s_n$.

$$\left[\text{Hint : First show that} \quad \frac{a_{N+1}}{s_{N+1}} + \cdots + \frac{a_{N+k}}{s_{N+k}} \geq 1 - \frac{s_N}{s_{N+k}}.\right]$$

(7 points)

(b) Repeat the question in part (a) with $\sum_{n=1}^{\infty} a_n/(1 + a_n\sqrt{n})$.
(8 points)

6. Give brief answers to the following questions:

$(5 \times 5 = 25 \text{ points})$

(a) Any connected metric space with more than one point is uncountable. True or false?

(b) There exists a metric space M such that there is no continuous map from M to any other metric space. True or false?

(c) Given any continuous function $f : [a, b] \to \mathbb{R}$, there exists $c \in [a, b]$ satisfying

$$\int_a^b f(t)\,dt = f(c)(b-a).$$

True or false?

(d) State the strong nested set property. Give an example of a metric space that does not have the strong nested set property. Provide reasons for your answer.

(e) Let $S = \{r_1 + r_2\pi : r_1, r_2 \in \mathbb{Q}\}$. Does there exist a function $f : \mathbb{R} \to \mathbb{R}$ whose set of discontinuities is $\mathbb{R} \setminus S$?