Be sure this exam has 12 pages including the cover

The University of British Columbia

Sessional Exams – 2013 Term 2
Mathematics 318 Probability with Physical Applications
Dr. G. Slade

Last Name: ____________________ **First Name:** ____________________

Student Number: ____

This exam consists of **7** questions worth **10** marks each,
1 question worth **3** marks, and **1** question worth **7** marks.
No aids are permitted. There are tables on the last page.
Please show all work and calculations. Numerical answers need not be simplified unless requested.

Problem	total possible	score
1.	10	
2.	10	
3.	10	
4.	10	
5.	10	
6.	10	
7.	10	
8.	3	
9.	7	
total	80	

(3 marks) 1. (a) Twelve different toys are to be divided among 3 children so that each child gets 4 toys. How many ways can this be done?

(3 marks) (b) How many ways can 4 men and 4 women sit in a row if no two men or two women sit next to each other?

(4 marks) (c) Suppose that n people, including Mr. Lee and Ms. Yang, are seated at random in a row of n seats. What is the probability that Mr. Lee and Ms. Yang sit next to each other?

(10 marks) 2. A woman has a brother with hemophilia but two parents who do not have the disease. Since hemophilia is caused by a recessive gene h on the X chromosome, we can infer than her mother is a carrier (i.e., the mother has the hemophilia gene h on one of her X chromosomes and the healthy gene H on the other), while her father has the healthy gene on his one X chromosome. Since the woman received one X chromosome from her father and one from her mother, there is a 50% chance that she is a carrier, and if so, there is a 50% chance that her sons will have the disease. If she has two sons without the disease, what is the probability that she is a carrier? (Give the actual number.)

3. A certain coin has probability p of showing Heads when flipped, and probability $1 - p$ of showing Tails, where $p \in (0, 1)$. When you are asked to name a distribution in this question, give all its parameters as well as the name.

(1 marks) (a) The coin is flipped repeatedly. Let X be the number of flips until the first Head is observed (counting the flip showing Heads). What is the distribution of X?

(1 marks) (b) The coin is flipped 1000 times. Let Y be the number of Heads observed in these 1000 flips. What is the distribution of Y?

(2 marks) (c) Suppose that $p = 0.005$, and that the coin is flipped 1000 times. The number of heads Y has the exact distribution given in part (b), but an approximation to this distribution is applicable here. Using this approximation, what is the (approximate) probability that exactly 5 Heads were observed?

(6 marks) (d) Suppose now that $p = \frac{1}{2}$ and that the coin is flipped 40,000 times. Let N be the number of Heads observed. What is the probability that $19,800 \leq N \leq 20,200$? (Give an actual number.)

4. (For this question you may find it useful to know that $\int_{-\infty}^{\infty} e^{itx^2} e^{-x^2/2} dx = \sqrt{2\pi}(1-2it)^{-1/2}$.)

(1 marks) (a) Let $X_1, X_2, X_3, \ldots$ be independent and identically distributed $\text{Exp}(\lambda)$ random variables. Let $S_n = X_1 + \cdots + X_n$. What is the distribution of S_n (name and parameter(s))?

(2 marks) (b) What is the characteristic function of S_n?

(4 marks) (c) Let $Z_1, Z_2, \ldots$ be independent and identically distributed standard normal random variables. Let $Y_n = Z_1^2 + \cdots + Z_n^2$. What is the characteristic function of Y_n?

(3 marks) (d) What is the distribution of Y_2? What is the distribution of Y_4? In both cases, give the name and parameter(s).

5. Buses pass my stop according to a Poisson process of rate 6 per hour. Thus the intervals between buses are independent Exp(6) random variables. I arrive at the bus stop at time zero.

(3 marks) (a) What is the probability that I wait at least 10 minutes for a bus?

(2 marks) (b) I wait ten minutes and no bus comes. What is the probability that I wait at least 10 additional minutes for a bus?

(2 marks) (c) Buses are independently full with probability $\frac{1}{3}$. As an experiment, I watch 12 buses pass. What is the probability that exactly 7 of them had space and 5 were full?

(3 marks) (d) Four buses arrive during the first half hour. What is the probability that exactly 6 buses arrive in total during the first hour?

6. Let d be a positive integer. Let $\vec{e}_i$ be the vector with d components whose i^{th} coordinate is one and whose other coordinates are zero. Consider the simple random walk on $\mathbb{Z}^d$ starting from $\vec{0} \in \mathbb{Z}^d$. The position of the walk after n steps is $\vec{S}_n = \vec{X}_1 + \ldots + \vec{X}_n$, where the $\vec{X}_i$ are i.i.d. and take the values $\pm\vec{e}_1, \ldots, \pm\vec{e}_d$ with equal probabilities $\frac{1}{2d}$.

(4 marks) (a) Compute the characteristic function $\phi_1(\vec{k})$ of $\vec{X}_1$.

(2 marks) (b) Expand $\phi_1(\vec{k})$ in a Taylor expansion about $\vec{k} = \vec{0}$, to second order.

(4 marks) (c) Prove that the random walk is recurrent in dimensions $d = 1, 2$ and transient in dimensions $d \geq 3$. You may use fact that the expected number of visits to $\vec{0}$ is given by the integral

$$\int_{[-\pi,\pi]^d} \frac{1}{1 - \phi_1(\vec{k})} \frac{d\vec{k}}{(2\pi)^d}.$$

7. Smith has four umbrellas which he leaves at home and at the casino. Each trip between the two locations, he takes an umbrella if it is raining and there is one at his starting location, and he never takes an umbrella if it is not raining. It rains independently each trip with probability p.

(3 marks) (a) Let X_n be the number of umbrellas at his current location at the start of the n^{th} trip. This defines a Markov Chain. Draw its transition diagram.

(4 marks) (b) Determine the stationary distribution of this Markov Chain.

(3 marks) (c) In the long run, what fraction of trips does Smith get wet?

(3 marks) 8. Suppose that whether it rains tomorrow depends on the weather yesterday and today. Specifically, suppose that if it rained both yesterday and today then the probability of rain tomorrow is 0.7; if it rained today but not yesterday then it will rain tomorrow with probability 0.5; if it rained yesterday but not today then it will rain tomorrow with probability 0.4; if it did not rain yesterday or today then it will rain tomorrow with probability 0.2. This can be formulated as a Markov Chain with state space:

state 0: it rained today and it rained yesterday,

state 1: it rained today but not yesterday,

state 2: it rained yesterday but not today,

state 3: it did not rain yesterday nor today.

What is the transition matrix for this Markov Chain?

9. Recall the "Ehrenfest coin chain" which is defined as follows. We consider M coins, each showing either Heads or Tails. At each time step, one of the coins is chosen uniformly at random and then flipped. The state of the chain is the number of Heads among the M coins, i.e., $X \in \{0, 1, \ldots, M\}$ denotes the state in which X coins are Heads and $M - X$ coins are Tails. Assume you are given an Octave function called nextStepEhr(X, M) which returns the (random) state at time 1 for the chain with M coins which is in state X at time 0.

(4 marks) (a) Write Octave code to generate variables $t_{M,i}$, for $1 \le M \le 15$ and $1 \le i \le 50$, all independent and defined by

$$t_{M,i} = \begin{array}{c}\text{number of steps it takes for a simulated chain with } M \\ \text{coins and started from state 0 to return to state 0.}\end{array}$$

(If you can't remember the syntax, explain in pseudocode.)

(3 marks) (b) For $1 \le M \le 15$, let $t_M = \frac{1}{50}\sum_{i=1}^{50} t_{M,i}$. What do you expect a plot of $\log t_M$ as a function of M to look like? Justify your answer.
Hint. Recall that the stationary distribution for the chain with M coins is $\mathrm{Bin}(M, \frac{1}{2})$.

Table 1: Common Distributions

Distribution	Mean	Variance	Characteristic function
Binomial (n,p)	np	$np(1-p)$	$(1-p+pe^{it})^n$
Geometric (p)	$1/p$	$\frac{1-p}{p^2}$	$\frac{pe^{it}}{1-(1-p)e^{it}}$
Poisson (λ)	λ	λ	$e^{\lambda(e^{it}-1)}$
Uniform (a,b)	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	$\frac{e^{ita}-e^{itb}}{it(b-a)}$
Exponential (λ)	$1/\lambda$	$1/\lambda^2$	$\frac{\lambda}{\lambda-it}$
Normal (μ,σ^2)	μ	σ^2	$e^{i\mu t-\sigma^2t^2/2}$

Table 2: Cumulative distribution function $\Phi(x)$ of standard Normal distribution

x	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990