The University of British Columbia
Math 318 — Probability with Physical Applications
2012, April 24

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First Name: ____________________ **Last Name:** ____________________

Student id. ______________________________ **Sorting:** __________

- This exam consists of **9 questions** worth a total of 100 points.
- Make sure this exam has 14 pages excluding this cover page.
- No aids except calculators are permitted.
- Duration: 2 hours 30 minutes.

Good luck, and enjoy your summer.

1. Each candidate should be prepared to produce his library/AMS card upon request.
2. Read and observe the following rules:
No candidate shall be permitted to enter the examination room after the expiration of one half hour, or to leave during the first half hour of the examination.
Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.
CAUTION - Candidates guilty of any of the following or similar practices shall be immediately dismissed from the examination and shall be liable to disciplinary action.
(a) Making use of any books, papers or memoranda, other than those authorized by the examiners.
(b) Speaking or communicating with other candidates.
(c) Purposely exposing written papers to the view of other candidates. The plea of accident or forgetfulness shall not be received.
3. Smoking is not permitted during examinations.

Question	Points	Score
1	12	
2	10	
3	16	
4	16	
5	12	
6	7	
7	12	
8	10	
9	5	
Total:	100	

Table 1: Common Distributions

Distribution	Mean	Variance	Characteristic function
Binomial (n,p)	np	$np(1-p)$	$(1-p+pe^{it})^n$
Geometric (p)	$1/p$	$\dfrac{1-p}{p^2}$	$\dfrac{pe^{it}}{1-(1-p)e^{it}}$
Poisson (λ)	λ	λ	$e^{\lambda(e^{it}-1)}$
Uniform (a,b)	$\dfrac{a+b}{2}$	$\dfrac{(b-a)^2}{12}$	$\dfrac{e^{ita}-e^{itb}}{it(b-a)}$
Exponential (λ)	$1/\lambda$	$1/\lambda^2$	$\dfrac{\lambda}{\lambda-it}$
Normal (μ,σ^2)	μ	σ^2	$e^{i\mu t-\sigma^2 t^2/2}$

The normal CDF:

x	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990

(12 marks) 1. We select randomly four cards out of a deck of 52 cards, (13 cards in each of 4 suits).

(a) What is the probability that the four cards are from four different suits?

(b) What is the probability that all four cards are from the same suit?

(c) What is the probability that there are no diamonds among the four cards?

(d) what is the expected number of different suits that are represented in the four cards?

(10 marks) 2. Suppose X and Y have joint p.m.f. as below (so $\mathbb{P}(X = 3, Y = 1) = 1/8$):

$X \backslash Y$	1	2	3
1	$1/4$	$1/6$	0
2	0	$1/8$	$1/6$
3	$1/8$	$1/6$	0

(a) Find the marginal distribution of X and that of Y.

(b) Compute $\mathbb{E}(X|Y = 2)$.

(c) Compute $\mathrm{Cov}(X, Y)$.

(16 marks) 3. Consider the Markov chain with states $\{1, 2, \ldots, 6\}$ and transition probability matrix

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/3 & 1/2 & 1/6 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/3 & 2/3 \\ 0 & 0 & 0 & 0 & 1/2 & 1/2 \end{pmatrix}$$

(a) Draw a transition diagram for this Markov chain.

(b) Determine the irreducible classes of this chain.

(c) Which states are recurrent and which are transient?

(d) Which states are periodic? With what period?

(e) If $X_0 = 2$, what is the probability that the chain reaches state 1?

(16 marks) 4. Each day, the blackboard is cleaned with probability $1/4$, independently of all other days. Let X_n be the number of days on day n since the board was last cleaned (so 0 means the board was cleaned today).

(a) Briefly explain why this is a Markov chain with states $\{0, 1, 2, \dots\}$ and transition probabilities $P_{n,n+1} = 3/4$ and $P_{n,0} = 1/4$ for $n \geq 0$.

(b) Is this Markov chain recurrent or transient?

(c) What is the stationary distribution for this Markov chain?

(d) In the long term, what fraction of time is spent at state 1?

(e) If the chain starts at $X_0 = 0$, let T be the next time it visits state 0. For example $\mathbb{P}(T = 1) = 1/4$. What is the distribution of T?

(f) What is $\mathbb{E}T$?

(12 marks) 5. A random variable X has p.d.f. $f(x) = ax^{-5}$ on $[b, \infty)$ and 0 on $(-\infty, b)$, for some $a, b > 0$.

(a) If $b = 1$, what is a?

(b) If $\mathbb{E}X = 1$ what are a and b?

(c) Let a, b be the ones you found in part (b). What is the variance of X?

(7 marks) 6. One hundred numbers are rounded to the nearest integer and then added. Using the central limit theorem, determine the probability that the sum of the rounded numbers will differ from the sum of the unrounded numbers by at most 3. You may assume that the roundoffs for the numbers are independent and uniformly distributed in $(-0.5, 0.5)$.

(12 marks) 7. Calls in English arrive at a call center according to a Poisson process with rate λ_e per hour. Calls in French arrive according to an independent Poisson process with rate λ_f. Let X_e and X_f be the number of English and French callers in one hour.

(a) What is the distribution of X_e? of X_f?

(b) What is the characteristic function of X_e? Of X_f?

(c) Let $X = X_e + X_f$ be the total number of calls in one hour. What is the characteristic function of X?

(d) What is the distribution of X? (Give the name and parameter).

(10 marks) 8. Consider the random walk S_n on $\mathbb{Z}$, with independent steps X_i, which take each of the values $\{-2, -1, 1, 2\}$ with probability $1/4$. (So $S_n = X_1 + \cdots + X_n$.)

(a) What is $\varphi(t)$: the characteristic function of X_i?

(b) Find the singularities of $\dfrac{1}{1-\varphi(t)}$ in $[-\pi, \pi]$.

(c) Prove that this random walk is recurrent. You may use that the expected number of visits to 0 is $\displaystyle\int_{-\pi}^{\pi} \frac{1}{1-\varphi(t)} dt$.

(5 marks) 9. A certain function `R()` returns a pair of random variables X, Y taking integer values. X and Y are not independent, but each time we call `R()` a new pair with the same joint distribution is generated. We use a computer simulation to estimate $\mathbb{E}(X|Y = 3)$, by using 10000 samples of the pair X, Y. The following code is incorrect. Explain the error, and suggest a way to fix it (the error is the same in both languages).

Python code:

```
accumulator = 0
for t in range(10000):
    X,Y = R()     # get a new sample
    if Y==3:
        accumulator += X

print accumulator/10000
```

Octave code:

```
accumulator = 0;
for t=1:10000
    X,Y = R();    % get a new sample
    if (Y==3)
        accumulator += X;
    end
end

accumulator/10000
```

Problem: ____________

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