Be sure this exam has 12 pages including the cover

The University of British Columbia

Sessional Exams – 2011 Term 2
Mathematics 318 Probability with Physical Applications
Dr. G. Slade

Last Name: ______________________ **First Name:** ______________________

Student Number: ____

This exam consists of **7** questions worth **10** marks each and **2** questions worth **5** marks each.
No aids are permitted.
There are tables on the last page.
Please show all work and calculations. Numerical answers need not be simplified.

Problem	total possible	score
1.	10	
2.	10	
3.	10	
4.	10	
5.	10	
6.	10	
7.	10	
8.	5	
9.	5	
total	80	

(5 points) 1. (a) Five dice are rolled simultaneously. Calculate the probability of rolling a full house (values a, a, a, b, b with a, b different).

(5 points) (b) A poker hand of five cards is dealt from a standard deck of 52 cards. Calculate the probability of a full house (face values a, a, a, b, b with a, b different).

2. In a certain lottery, a ticket is a winning ticket with probability $1/100$. In the following, explain your answers.

(2 points) (a) What is the expected number of tickets until the first winning ticket?

(2 points) (b) 500 tickets are sold. What is the expected number of winning tickets?

(3 points) (c) 500 tickets are sold. What is the probability that the number of winning tickets is equal to the expected number of winning tickets?

(3 points) (d) How many tickets must be purchased to ensure that the probability is approximately $1/2$ that they include at least one winning ticket?

3. The occurrence of major earthquakes in a certain location can be modelled by a Poisson process with rate $\lambda = \frac{1}{100}$ per year. The time intervals between major earthquakes are thus given by independent exponential random variables with $\lambda = \frac{1}{100}$.

(3 points) (a) What is the probability that the next major earthquake occurs in less than 50 years?

(4 points) (b) Let N_{500} denote the number of major earthquakes that occur during the next 500 years. What kind of random variable is N_{500}? What is its mean?

(3 points) (c) Compute the probability that there are three or fewer major earthquakes in the next 500 years.

(10 points) 4. A binary message—either 0 or 1—must be transmitted by wire from location A to location B. However, data sent over the wire are subject to channel noise disturbance, so to reduce the possibility of error, the value 2 is sent if the message is 1 and the value -1 is sent if the message is 0. If x is the value sent at A ($x = -1$ or $x = 2$), then the value received at B is $x + N$, where N represents the noise. Assume that N is a normal random variable with mean $\mu = 0$ and variance $\sigma^2 = 0.25$. Assume also that the message to be transmitted is equally likely to be either 0 or 1. When the message is received at B the receiver decodes it according to the following rule:

If $R \geq 0.5$, then 1 is concluded.

If $R < 0.5$, then 0 is concluded.

The message concluded at B is 1. What is the probability that the message was incorrectly transmitted?

(10 points) 5. An astronomer is interested in measuring, in light years, the distance from her observatory to a distant star. Although she has a measuring technique, she knows that because of changing atmospheric conditions and experimental error, each time a measurement is made it will not yield the exact value but rather an approximate value. As a result the astronomer plans to make a series of measurements and then use the average of these as the estimated value of the actual distance. If the astronomer believes that the values of the measurement errors are not systematic and are described by a random variable with mean d (the true distance) and a variance of 4 light years, use the central limit theorem to determine approximately the number of measurements that should be made to be 95% sure that the estimated distance is accurate to within ± 0.5 light years.

6. Consider a random walk on the 2-dimensional square lattice that takes each of the four steps $(+1,+1)$, $(+1,-1)$, $(-1,+1)$, $(-1,-1)$ with equal probabilities $\frac{1}{4}$.

(4 points) (a) Calculate the characteristic function $\phi_1(k_1,k_2)$ of a single step. Simplify your answer as much as possible. (Recall the trigonometric identity $\cos(x+y)+\cos(x-y) = 2\cos x \cos y$.)

(3 points) (b) Identify all singularities of $1/(1-\phi_1(k_1,k_2))$, for $k_1,k_2 \in [-\pi,\pi]$.

(3 points) (c) Is the random walk transient or recurrent? Explain in detail.

7. Two white balls and two black balls are distributed in two urns in such a way that each urn contains two balls. We say that the system is in state i, $(i = 0, 1, 2)$ if the first urn contains i white balls. At each step, we draw one ball from each urn, place the ball drawn from the first urn in the second urn, and place the ball drawn from the second urn in the first urn. Let X_n denote the state of the system after the nth step. This defines a Markov chain.

(4 points) (a) Calculate the one-step transition matrix for this Markov chain.

(1 points) (b) Calculate the two-step transition matrix.

(4 points) (c) Compute the stationary distribution for the chain.

(1 points) (d) In the long run, what fraction of time does the first urn contain two white balls?

8. Fix a number p with $0 < p < 1$. Consider the Markov chain on the non-negative integers $\{0, 1, 2, \ldots\}$ whose transition probabilities are given by

$$P_{n,n+1} = p, \quad P_{n,0} = 1 - p, \quad \text{for all } n \geq 0.$$

Suppose that the Markov chain is initially in state 0, and let T_0 denote the time of first return to 0 (i.e., T_0 is the smallest value of $n > 0$ such that $X_n = 0$, if such a value exists, and otherwise $T_0 = \infty$).

(3 points) (a) Determine the probability mass function of T_0.

(2 points) (b) Determine the expected value ET_0.

(5 points) 9. We wish to use Octave to compute the integral $\int_0^2 e^{-3x/2}\,dx$ via Monte Carlo integration. In general, for $\int_a^b f(x)dx$, we simulate a large number of Unif(a,b) random numbers $U_1,\dots,U_n$ and use the approximation

$$\int_a^b f(x)dx \approx (f(U_1)+\cdots+f(U_n))\,\frac{b-a}{n}.$$

The code below runs, but gives incorrect answers:

```
% Script to perform the required integral

% Generate a number of uniform numbers
N = 10000;
uniforms = unifrnd(0,2,N,1); % generates a N x 1 matrix of
                             % Unif[0,2] random numbers

accumulator = 0;

for i = 1:N
  accumulator = accumulator + uniforms(i);
end % or endfor; both work

% Compute the approximate value of the integral
approximation  = exp((-3/2)*(accumulator/N));
```

The exact value is $\int_0^2 e^{-3x/2}\,dx = 2(1-e^{-3})/3 \approx 0.63348$, but running the above code four times gave the answers 0.22423, 0.22461, 0.22568, and 0.22329.

Find the problems in the code and state how to correct it. If you find it easier to rewrite all or some of the code, feel free to do so. (If you can't remember the syntax, explain what you are trying to do in pseudocode.)

Table 1: Common Distributions

Distribution	Mean	Variance	Characteristic function
Binomial (n, p)	np	$np(1-p)$	$(1-p+pe^{it})^n$
Geometric (p)	$1/p$	$\frac{1-p}{p^2}$	$\frac{pe^{it}}{1-(1-p)e^{it}}$
Poisson (λ)	λ	λ	$e^{\lambda(e^{it}-1)}$
Uniform (a, b)	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	$\frac{e^{ita}-e^{itb}}{it(b-a)}$
Exponential (λ)	$1/\lambda$	$1/\lambda^2$	$\frac{\lambda}{\lambda-it}$
Normal (μ, σ^2)	μ	σ^2	$e^{i\mu t-\sigma^2t^2/2}$

Table 2: Cumulative distribution function $\Phi(x)$ of standard Normal distribution

x	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990