Be sure this exam has 13 pages including the cover

The University of British Columbia

Sessional Exams – 2010 Term 2
Mathematics 318 Probability with Physical Applications
Dr. G. Slade

Last Name: _____ **First Name:** _____

Student Number: _____

This exam consists of **8** questions worth **10** marks each. No aids are permitted.
There are tables on the last two pages.
Please show all work and calculations. Numerical answers need not be simplified.

Problem	total possible	score
1.	10	
2.	10	
3.	10	
4.	10	
5.	10	
6.	10	
7.	10	
8.	10	
total	80	

(1 points)   1.   (a) Define the moment generating function of a random variable X.

(3 points)   (b) Derive the moment generating function of a geometric random variable with parameter p, starting from the definition of moment generating function.

(3 points)   (c) A fair coin is tossed until the nth Head appears. Find the moment generating function for the total number of tosses.

(3 points)   (d) Determine the variance of the total number of tosses, in part (c).

2. Data indicate that the number of traffic accidents in Vancouver on a rainy day is a Poisson random variable with mean 9, whereas on a dry day it is a Poisson random variable with mean 3. If it will rain tomorrow with probability 0.6, find:

(4 points)   (a) the probability that there are no accidents tomorrow

(3 points)   (b) the expected number of accidents tomorrow

(3 points)   (c) the variance of the number of accidents tomorrow.

(10 points) 3. A plane is missing, and it is presumed that it was equally likely to have gone down in any of 3 different regions. Due to the geographical and environmental conditions of the regions, the probability that the plane will be found upon a search of the ith region when the plane is, in fact, in that region is $\frac{1}{4}$ in region 1, $\frac{1}{2}$ in region 2, and $\frac{3}{4}$ in region 3 What is the conditional probability that the plane is in the ith region, given that a search of region 1 is unsuccessful, $i = 1, 2, 3$?

4. The lifetime (in hours) of a certain electronic component is an exponential random variable with parameter $\lambda = \frac{1}{4}$, and thus has probability density function $f(x) = \frac{1}{4}e^{-x/4}$ for $x \geq 0$. As soon as a component fails, it is replaced immediately, so that there is always a working component.

(3 points)   (a) Find the probability that a component's lifetime exceeds 6 hours.

(3 points)   (b) Let Y denote the time that the fourth component is installed. What kind of random variable is Y, i.e. what distribution does Y have?

(4 points)   (c) The first component is still working after 6 hours. What is the probability that exactly four components have failed within 24 hours of the installation of the first component?

5. A certain scientific theory supposes that mistakes in cell division occur according to a Poisson process with rate 1.25 per year, and that an individual dies when 100 such mistakes have occurred. Assuming this theory, find:

(2 points)   (a) the mean lifetime of an individual.

(2 points)   (b) the variance of the lifetime of an individual.

(3 points)   (c) the approximate probability that an individual reaches age 90 (use the central limit
                      theorem).

(3 points)   (d) the approximate conditional probability that an individual reaches age 100, given that
                      the individual has reached age 90 (use the central limit theorem).

6. Consider simple random walk on Z^3, taking steps $(\pm1,0,0)$, $(0,\pm1,0)$, $(0,0,\pm1)$ with probabilities $1/6$.

(5 points)   (a) Write down a formula for the probability p_{2n} that the walk starting at the origin returns to the origin in $2n$ steps. Explain how you obtained your formula.

(5 points)   (b) It is possible to show (you need not show this) that

$$p_{2n} \leq \frac{1}{2^{2n}} \binom{2n}{n} \frac{1}{3^n} \frac{n!}{(\lfloor n/3 \rfloor!)^3},$$

where $\lfloor n/3 \rfloor$ denotes the greatest integer less than or equal to $n/3$. Using this, show that the walk is transient. (Recall Stirling's formula $n! \sim n^n e^{-n} \sqrt{2\pi n}$.)

7. At all times, an urn contains N balls, some black and some white. At each stage, a coin having probability $p \in (0, 1)$ of landing heads is flipped. If heads appears, then a ball is chosen at random from the urn and is replaced by a white ball. If tails appears, then a ball is chosen at random from the urn and is replaced by a black ball. Let X_n denote the number of white balls in the urn after the nth stage.

(3 points)   (a) What are the transition probabilities of this chain?

(1 points)   (b) Explain why the Markov chain has a unique stationary distribution.

(3 points)   (c) Find the stationary distribution.

(3 points)   (d) Show that the chain is reversible.

8. Each day, one of 3 possible items is requested, the ith one with probability $\frac{i}{6}$ $(i = 1, 2, 3)$. These items are at all times arranged in an ordered list which is revised as follows: The item selected is moved to the top of the list with the relative positions of the other two elements left unchanged. The state at any time is the current list ordering.

(5 points)   (a) Determine the transition matrix of the above Markov chain.

(5 points)   (b) The following code is intended to find the mean return time of each state in the Markov chain in part (b) using Octave, but it has two bugs in it. The matrix P has been initialized to equal the matrix of part (a) (this is not explicitly written below but you should assume this step has been prepared). Find the two bugs and fix the errors. If you do not remember the syntax for how to fix the code, write pseudo-code explaining what you would do.

Note: there are no syntactical errors in this code. That is, if you entered the code into Octave as written, it would run but produce an incorrect result.

Note: [eigenvectors, eigenvalues] = eig(P) puts right eigenvectors of P in the columns of eigenvectors and puts the corresponding eigenvalues in the diagonal of the square matrix eigenvalues. In the code below, assume that the eigenvalue 1 is located in the top left entry of the matrix eigenvalues.

```
% Find the eigenvectors and eigenvalues of P
[eigenvectors, eigenvalues] = eig(P)

% The eigenvalue 1 is in the first column; thus the eigenvector
% corresponding to it is also in the first column.
% This following line picks it out.
unnormalized = eigenvectors(:,1);

% Calculate the mean return time
sum_of_eigenvector = 0;
for i=1:3
  sum_of_eigenvector = sum_of_eigenvector + unnormalized(i);
end

stationary = unnormalized/sum_of_eigenvector;

mean_return_time = stationary
```

Table 1: Common Distributions

Distribution	Mean	Variance	Characteristic function
Binomial (n, p)	np	$np(1-p)$	$(1 - p + pe^{it})^n$
Geometric (p)	$1/p$	$\dfrac{1-p}{p^2}$	$\dfrac{pe^{it}}{1 - (1-p)e^{it}}$
Poisson (λ)	λ	λ	$e^{\lambda(e^{it}-1)}$
Uniform (a, b)	$\dfrac{a+b}{2}$	$\dfrac{(b-a)^2}{12}$	$\dfrac{e^{ita} - e^{itb}}{it(b-a)}$
Exponential (λ)	$1/\lambda$	$1/\lambda^2$	$\dfrac{\lambda}{\lambda - it}$
Normal (μ, σ^2)	μ	σ^2	$e^{i\mu t - \sigma^2 t^2/2}$

Table 2: Cumulative distribution function $\Phi(x)$ of standard Normal distribution

x	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990