The University of British Columbia

Final Examination - April 23, 2008

Mathematics 316

Section 201

Instructor: Anmar Khadra

Closed book examination Time: 2.5 hours

Name ______________________ **Signature** ______________________

Student Number ______________

Special Instructions:

- Be sure that this examination has 13 pages. Write your name on top of each page.
- Two formula sheets are provided.
- No programmable/graphing calculators or notes are permitted.
- In case of an exam disruption such as a fire alarm, leave the exam papers in the room and exit quickly and quietly to a pre-designated location.

Rules governing examinations

- Each candidate should be prepared to produce her/his library/AMS card upon request.
- No candidate shall be permitted to enter the examination room after the expiration of one half hour, or to leave during the first half hour of examination.
- Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.
- CAUTION - Candidates guilty of any of the following or similar practices shall be immediately dismissed from the examination and shall be liable to disciplinary action.
  (a) Making use of any books, papers, or memoranda, other than those authorized by the examiners.
  (b) Speaking or communicating with other candidates.
  (c) Purposely exposing written papers to the view of other candidates.
- Smoking is not permitted during examinations.

1		20
2		9
3		15
4		16
5		20
6		20
Total		100

[20] 1. Consider the following differential equation

$$xy'' - xy' + y = 0.$$

(a) Show that $x_0 = 0$ is a regular singular point of the equation.

(b) Use Frobenius' method to obtain at least one series solution centered at $x_0 = 0$. Explain how your result from using Frobenius' method is consistent with the indicial roots derived in your solution.

(c) Hence derive the general solution of the differential equation.

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[9] 2. Use the eigenfunction-expansion method to solve

$$\nabla^2 u = xy + u$$

inside the unit square ($0 < x < 1$, $0 < y < 1$), given that u is zero on the boundary.

[15] 3. Use the method of separation of variables to solve the wave equation on a unit disk, given by

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = 100\left(\dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r}\right) \\ u(1,t) = 0 \\ u(r,0) = f(r), \quad \dfrac{\partial u}{\partial t}(r,0) = 0, \quad 0 \leq r < 1. \end{cases}$$

What type of symmetry is maintained by this problem?

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[16] 4. Consider the following Sturm-Liouville problem, given by

$$\left(x^{-1}y'\right)' + \lambda x^{-3} y = 0, \quad 1 < x < 2, \quad y(1) = y(2) = 0. \tag{1}$$

(a) Determine if (1) is a regular or singular Sturm-Liouville problem.

(b) Find the eigenvalues and eigenfunctions of equation (1). (Hint: transform equation (1) into Euler's equation.)

(c) Express the orthogonality relations between two eigenfunctions corresponding to two different eigenvalues obtained in part (b).

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[20] 5. Consider the following IBVP

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} - x \\ u(0,t) = 0, \; u(1,t) = \dfrac{1}{6} \\ u(x,0) = \dfrac{7}{6}x^3, \quad 0 < x < 1. \end{cases}$$

(a) Find the steady state solution, $v(x)$, associated with the above problem.

(b) Derive the IBVP satisfied by the function $w(x,t) := u(x,t) - v(x)$ and solve it.

(c) Use (a) and (b) to find $u(x,t)$.

(d) Notice that, according to the initial condition, $u(1, 0^-) = 7/6$, while according to the second BC, $u(1,0) = 1/6$. Does the solution in (c) converge uniformly at $t = 0$? Explain.

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[20] 6. Solve the IBVP

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + e^{-t}x \\ u(0,t) = u(1,t) = 0 \\ u(x,0) = \sin(\pi x), \quad 0 < x < 1. \end{cases}$$

Determine the fundamental mode and the long term behaviour of the solution.

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Important Integrals

$$\int x\sin(n\pi x)dx = \frac{-x\cos(n\pi x)}{n\pi} + \frac{\sin(n\pi x)}{n^2\pi^2} + c.$$

$$\int x^3\sin(n\pi x)dx = \frac{-x^3\cos(n\pi x)}{n\pi} + \frac{6x\cos(n\pi x)}{n^3\pi^3} + \frac{(3n^2\pi^2x^2 - 6)\sin(n\pi x)}{n^4\pi^4} + c.$$

Double Fourier Half Range Sine Series Expansion

$$f(x,y) = \sum_{n,m=1}^{\infty} A_{mn}\sin(\frac{m\pi}{a}x)\sin(\frac{n\pi}{b}y)$$

$$A_{mn} = \frac{4}{ab}\int_0^a\int_0^b f(x,y)\sin(\frac{m\pi}{a}x)\sin(\frac{n\pi}{b}y)dydx$$

List of Properties and Identities for Bessel Functions

Let m be a nonnegative integer and ν be a nonnegative real number.

1. $J_{-m}(x) = (-1)^m J_m(x)$.
2. $J_m(-x) = (-1)^m J_m(x)$. i.e., $J_m(x)$ is an even function when m is even and it is an odd function when m is odd.
3. $J_m(0) = \begin{cases} 0 & m > 0 \\ 1 & m = 0. \end{cases}$
4. $\lim_{x\to 0^+} Y_m(x) = -\infty$.
5. $xJ'_\nu(x) = \nu J_\nu(x) - xJ_{\nu+1}(x)$.
6. $xJ'_\nu(x) = -\nu J_\nu(x) + xJ_{\nu-1}(x)$.
7. $J_{\nu-1}(x) - J_{\nu+1}(x) = 2J'_\nu(x)$.
8. $J_{\nu-1}(x) + J_{\nu+1}(x) = \frac{2\nu}{x}J_\nu(x)$.
9. $\frac{d}{dx}\left[x^\nu J_\nu(x)\right] = x^\nu J_{\nu-1}(x)$.
10. $\frac{d}{dx}\left[x^{-\nu} J_\nu(x)\right] = -x^{-\nu} J_{\nu+1}(x)$.
11. $\int x^{\nu+1}J_\nu(x)dx = x^{\nu+1}J_{\nu+1}(x) + c$.
12. $\int x^{-\nu+1}J_\nu(x)dx = -x^{-\nu+1}J_{\nu-1}(x) + c$.