THE UNIVERSITY OF BRITISH COLUMBIA
SESSIONAL EXAMINATIONS - APRIL 2009
MATHEMATICS 310

This examination consists of 2 pages.
Check to ensure that this paper is complete.

TIME: 2 1/2 hours

1. [15 marks] In the vector space $V = P_4(\mathbb{R})$ of polynomials with real coefficients of degree less than or equal to 4, determine whether the following subsets are subspaces. Find the dimension of each that is a subspace.
   a) $W = \{f \in V \mid f(0) + f'(0) = 0\}$ (f' denotes the derivative of f)
   b) $W = \{f \in V \mid f(0) + f'(0) = 1\}$

2. [15 marks] Let V be the 3-dimensional vector space of real-valued functions of the form $ae^x + be^{2x} + ce^{3x}$. Define $T : V \to V$ by $T(f(x)) = f(x) + f'(x)$ (where f' is the derivative of f).
   a) Show that T is a linear transformation.
   b) One basis of V is e^x, e^{2x}, e^{3x}. Show that $e^x + e^{2x}, e^{2x} + e^{3x}, e^x + e^{3x}$ is another basis, and find the matrix of T with respect to this second basis.

3. [15 marks] Let A be the 3×3 matrix

$$A = \begin{pmatrix} -1 & -2 & -4 \\ 1 & 2 & 2 \\ 1 & 1 & 3 \end{pmatrix}.$$

   a) Find the determinant and the trace of A.
   b) A has only 2 distinct eigenvalues, one of which equals 1. Find the other eigenvalue of A.
   c) Show that A is diagonalizable. What is the minimum polynomial of A?

4. [15 marks] Let V be the vector space $M_{2\times 2}(\mathbb{R})$ whose elements are 2×2 real matrices. Define a linear transformation T from V to V by $T(A) = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix} \cdot (A - A^t)$, where A^t denotes the transpose of A (you may assume that T is indeed a linear transformation). Find a Jordan canonical form of T.

5. [15 marks] On the vector space $V = P_1(\mathbb{R})$ define an inner product $\langle f, g\rangle = \int_{-1}^{1} f(t)g(t)\, dt$ (you do not need to prove that this is an inner product).
   a) Apply the Gram-Schmidt process to the basis $1, t$ of V to produce an orthonormal basis of V.
   b) Let T be the linear transformation of V defined by $T(f) = f' + 3f$ (you may assume that T is indeed a linear transformation). Let T^* be the adjoint of T with respect to the given inner product. Evaluate $T^*(4 - 2t)$.

6. [9 marks] Let A be an invertible real $n \times n$ matrix.

   a) Prove that 0 is not an eigenvalue of A.

   b) If λ is an eigenvalue of A, prove that λ^{-1} is an eigenvalue of A^{-1}.

7. [16 marks] Determine, with justification, whether the following statements are true or false.

   a) If A and B are symmetric 3×3 matrices such that $AB = 0$, then $BA = 0$.

   b) The 4×4 matrix all of whose entries equal 1 except for its 4,4 entry which equals 2 is diagonalizable.

   c) There are 5×5 matrices A and B of rank 4 and 3 respectively such that AB has rank 4.

   d) There are 5×5 matrices A and B of rank 4 and 3 respectively such that AB has rank 2.