**Be sure this exam has 12 pages including the cover**

**The University of British Columbia**

**MATH 308, Section 101, Instructor Tai-Peng Tsai**

**Final Exam – December 2011**

**Family Name** ______________________ **Given Name** ______________________

**Student Number** ______________________ **Signature** ______________________

**No notes nor calculators.**

**Rules Governing Formal Examinations**:

1. Each candidate must be prepared to produce, upon request, a UBCcard for identification;

2. Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions;

3. No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination;

4. Candidates suspected of any of the following, or similar, dishonest practices shall be immediately dismissed from the examination and shall be liable to disciplinary action;
(a) Having at the place of writing any books, papers or memoranda, calculators, computers, sound or image players/recorders/transmitters (including telephones), or other memory aid devices, other than those authorized by the examiners;
(b) Speaking or communicating with other candidates;
(c) Purposely exposing written papers to the view of other candidates or imaging devices. The plea of accident or forgetfulness shall not be received;

5. Candidates must not destroy or mutilate any examination material; must hand in all examination papers, and must not take any examination material from the examination room without permission of the invigilator; and

6. Candidates must follow any additional examination rules or directions communicated by the instructor or invigilator.

| problem | max | score |
|---|---|---|
| 1. | 10 | |
| 2. | 10 | |
| 3. | 10 | |
| 4. | 15 | |
| 5. | 10 | |
| 6. | 10 | |
| 7. | 10 | |
| 8. | 15 | |
| 9. | 10 | |
| total | 100 | |

1. Suppose a conic has two focus points $F(4, 0)$ and $F'(-4, 0)$.

(5 pt) (a) Find the equation of the conic if a directrix has the equation $x = 9$.

(5 pt) (b) Find the equation of the conic if the conic has eccentricity $e = 2$.

(5 pt) 2. (a) Find the equation of the tangent line to the parabola $x = y^2$ at the point $(1, -1)$.

(5 pt) (b) Classify the conic $x^2 - 6xy + 9y^2 = 1$.

3. The triangle $\triangle ABC$ has vertices $A(-1, 2)$, $B(-3, -1)$ and $C(3, 1)$, and the points $P(1, \frac{1}{3})$, $Q(1, \frac{3}{2})$, and $R(-\frac{5}{3}, 1)$ lie on $BC$, $CA$ and $AB$, respectively.

(7 pt) (a) Determine the ratios in which $P$, $Q$ and $R$ divide the sides of the triangle.

(3 pt) (b) Determine whether or not the lines $AP$, $BQ$ and $CR$ are concurrent.

(5 pt) 4. (a) Find the Euclidean transformation which is the counterclockwise rotation about the point $(2, 2)$ by angle $\frac{\pi}{3}$.

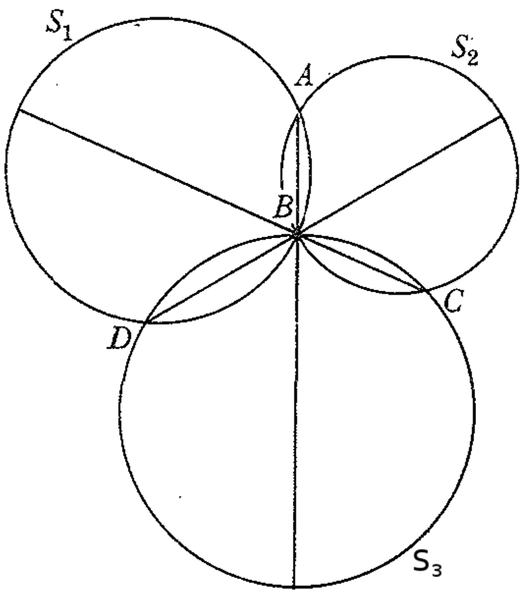
(5 pt) (b) Find the affine transformation which maps the points $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 4 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ to the points $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $\begin{pmatrix} 5 \\ -4 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$, respectively.

(5 pt) (c) Find the Möbius transformation which maps the points $1, 2, i$ to $1, 2, 3$.

(5 pt) 5. (a) Find the image of the line $3x + y = 4$ under the affine transformation $t(\mathbf{x}) = A\mathbf{x} + \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ with $A = \begin{pmatrix} 4 & 5 \\ 1 & 1 \end{pmatrix}$.

(5 pt) (b) Find the image of the circle $|z| = 1$ under the Möbius transformation $M(z) = \dfrac{iz+1}{z+i}$.

(10 pt) 6. Let two circles $S_1$ and $S_2$ intersect in $A$ and $B$, and let the diameters of $S_1$ and $S_2$ through $B$ cut $S_2$ and $S_1$ in $C$ and $D$. Let $S_3$ be the circle containing $B$, $C$, and $D$. Show that line $\overleftrightarrow{AB}$ passes through the center of circle $S_3$.



(10 pt) 7. The circles $C_1$ and $C_2$ in an Apollonian family of circles have the segments $[-11, -6]$ and $[0, 16]$, respectively, of the $x$-axis as diameters. Determine the point circles of the family.

(7 pt) 8. (a) Find the $d$-line that passes through the two $d$-points $i/3$ and $-1/3$, and sketch it.

(8 pt) (b) Find the Euclidean radius of the non-Euclidean circle $C = \{z \in \mathscr{D} : \quad d(\frac{i}{3}, z) = \frac{1}{3}\}$.
Do not evaluate or simplify the solution. Note $\ln 2 \approx 0.6931$ and $\ln 3 \approx 1.0986$.

(10 pt) 9. Answer YES or NO to the following questions. Anything else will not earn any partial credits. Answering both YES and NO earns 0.

(a) Is it true that an affine transformation preserves ratio of lengths on parallel lines?

(b) Is it true that, for any point $P$ inside triangle $\triangle ABC$, there is an affine transformation $t$ such that $t(P)$ is outside of triangle $\triangle t(A)t(B)t(C)$?

(c) Is it true that an ellipse can be mapped onto a circle by an affine transformation?

(d) Is it true that an ellipse can be mapped onto a circle by an inversive transformation?

(e) Is it true that an inversion is a Möbius transformation?

**Some Formulas**

| | **eccentricity** | **standard eq** | **focus** | **directrix** | **tangent at** $(x_1, y_1)$ |
|---|---|---|---|---|---|
| ellipse | $0 \le e < 1$ | $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ | $(\pm ae, 0)$ | $x = \pm a/e$ | $\frac{x_1 x}{a^2} + \frac{y_1 y}{b^2} = 1$ |
| hyperbola | $e > 1$ | $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ | $(\pm ae, 0)$ | $x = \pm a/e$ | $\frac{x_1 x}{a^2} - \frac{y_1 y}{b^2} = 1$ |
| parabola | $e = 1$ | $y^2 = 4ax$ | $(a, 0)$ | $x = -a$ | $y_1 y = 2a(x + x_1)$ |

Above $b = a\sqrt{|1 - e^2|}$

1. Rotation about the origin by angle $\theta$ is $t\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$, or $t(z) = e^{i\theta}z$.
2. Reflection about the line passing the origin with angle $\theta/2$ to the $x$-axis is $t\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$, or $t(z) = e^{i\theta}\bar{z}$.
3. The affine transformation of $\mathbb{R}^2$ mapping $\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$ to $P = \left[\begin{smallmatrix} p_1 \\ p_2 \end{smallmatrix}\right]$, $Q = \left[\begin{smallmatrix} q_1 \\ q_2 \end{smallmatrix}\right]$, and $R = \left[\begin{smallmatrix} r_1 \\ r_2 \end{smallmatrix}\right]$ is
$$t(\vec{x}) = \vec{P} + \mathbf{A}\vec{x}, \quad \mathbf{A} = (\vec{PQ}|\vec{PR}) = \begin{pmatrix} q_1 - p_1 & r_1 - p_1 \\ q_2 - p_2 & r_2 - p_2 \end{pmatrix}.$$
4. Affine transformations preserve collinearity, coincidence, parallel lines, and signed ratio of lengths along the same direction.
5. Ceva's and Menelaus' theorems: $P \in BC$, $Q \in CA$, $R \in AB$, $\frac{AR}{RB} \cdot \frac{BP}{PC} \cdot \frac{CQ}{QA} = \pm 1$ implies coincidence or collinearity.
6. Inversion in the unit circle: $(x, y) \mapsto (\frac{x}{r^2}, \frac{y}{r^2})$ where $r = \sqrt{x^2 + y^2}$. Inversion in the circle $|z - m| = r$ is $z \mapsto \frac{r^2}{\bar{z} - \bar{m}} + m$.
7. The stereographic projection $\pi : \mathbb{S}^2 \to \mathbb{C}$ satisfies $\pi : (X, Y, Z) \mapsto \frac{X + iY}{1 - Z}$ with $\pi^{-1} : x + iy \mapsto \frac{1}{x^2 + y^2 + 1}(2x, 2y, x^2 + y^2 - 1)$.
8. Inversive transformations on $\hat{\mathbb{C}}$ are of the form $t(z) = M(z)$ or $t(z) = M(\bar{z})$ with $M(z) = \frac{az + b}{cz + d}$, $ad \ne bc$. Isometries correspond to $M(z) = az + b$ with $|a| = 1$.
9. Non-Euclidean transformations on $\mathscr{D}$ are $t(z) = M(z)$ or $t(z) = M(\bar{z})$ with $M(z) = \frac{az + b}{\bar{b}z + \bar{a}}$, $|b| < |a|$. An inversion in the d-line $|z - \alpha| = r$ with $|\alpha|^2 = r^2 + 1$ is $t(z) = \frac{\alpha\bar{z} - 1}{\bar{z} - \bar{\alpha}}$.
10. The distance function on $\mathscr{D}$ is $d(z_1, z_2) = \tanh^{-1}(\frac{|z_1 - z_2|}{|1 - \bar{z}_1 z_2|})$ and $d(z, 0) = \tanh^{-1}(|z|)$. Note $y = \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$ and $x = \tanh^{-1} y = \frac{1}{2}\ln\frac{1 + y}{1 - y}$ for $0 \le y < 1$.