The University of British Columbia

Final Examination - April 17, 2009

Mathematics 308

Section 201 – D. Rolfsen, Instructor

Closed book examination Time: 2.5 hours

Last Name ____________________ **First** _____________ **Signature** __________________

Student Number ____________________

Special Instructions:

No books, notes, or calculators are allowed. Unless it is otherwise specified, answers may be left in "calculator-ready" form, where calculator means basic scientific calculator. Show all your work, little or no credit will be given for a numerical answer without the correct accompanying work. If you need more space than the space provided, use the back of the previous page. Where boxes are provided for answers, put your final answers in them.

Rules governing examinations

1		6
2		10
3		24
4		8
5		8
6		6
7		6
8		10
9		10
10		6
11		6
Total		100

Marks

[6] 1. What is the interior angle of a regular 7-gon, measured in radians?

Answer:

[10] 2. (a) Write the complex number $5e^{-3\pi i/4}$ in the form $x + iy$:

Answer:

(b) Give all complex solutions Z to the equation $Z^2 = 1 - i$. (You may write your answers in polar form or the form $x + iy$.)

Answer:

3. True or false? Explain.

[3] (a) For any two vectors X and Y in $\mathbb{R}^n$ we have $|X| + |Y| \leq |X - Y|$.

True/False?

[3] (b) In any triangle, the centroid, circumcentre, and orthocentre are collinear.

True/False?

[3] (c) The inequality $|x| + |y| + |z| \leq 1$ defines a regular octahedron in $\mathbb{R}^3$.

True/False?

[3] (d) The circumcentre of a triangle lies outside the triangle if one of the triangle's angles is greater than a right angle.

True/False?

[3] (e) If four distinct complex numbers do not lie in a straight line or circle, then their cross-ratio cannot be real.

True/False?

[3] (f) There are exactly 48 isometries of the regular dodecahedron (including the identity).

True/False?

[3] (g) If Q and Q' are quaternions, the conjugate of QQ' is equal to $\bar{Q}\bar{Q}'$.

True/False?

[3] (h) If Q is a pure imaginary quaternion and $Q \neq 0$, then Q^{-1} is also pure imaginary.

True/False?

[8] 4. Consider the triangle ABC in $\mathbb{R}^2$ whose vertices A, B, C have coordinates $(0,1), (t,0)$ and $(s,0)$, respectively.

(a) Denote the medians of ABC, as usual, by AA', BB' and CC'. Find the coordinates of A' and of the centroid G.

$A' =$

$G =$

(b) Let the altitudes of ABC be denoted AD, BE and CF, respectively. Find the coordinates of D and of the orthocentre H.

$D =$

$H =$

[8] 5. Consider the Möbius transformation $T(Z) = \dfrac{iZ}{1+Z}$.

(a) What is the image of the unit circle under T?

Answer:

(b) What points are fixed by T?

Answer:

(c) Give a formula for the inverse transformation T^{-1} (meaning $T^{-1}(T(Z)) = Z$)

Answer:

(d) What is the *inverse* image of the unit disk under T?

Answer:

[6] 6. Find a Möbius transformation $T(Z) = \dfrac{AZ+B}{CZ+D}$ satisfying:

$$T(0) = 0, \quad T(1) = 1, \quad T(i) = \infty.$$

Answer:

7. Suppose C and C' are intersecting circles in the complex plane, and R and R' are inversions (a.k.a. reflections) in C and C', respectively. If $R(C') = C'$, does it follow that $R'(C) = C$?
[6] Explain.

Answer:

[10] 8. The quaternion $Q = 5 + 4i - 3k$ has a polar form $Q = r(\cos\theta + H\sin\theta)$, where $H^2 = -1$. Determine the value of r, θ and H.

$r =$

$\theta =$

$H =$

Calculate Q^2 and the inverse of Q.

$Q^2 =$

$Q^{-1} =$

[10] 9. If Q is a unit quaternion, define $M_Q(X) = QX\bar{Q}$, where X is a quaternion variable.

(a) Find the value of Q so that the action of M_Q on the imaginary quaternions

$$\mathbb{R}^3 = \{xi + yj + zk\}$$

consists of $\pi/3$ rotation about the line defined by the equations $x = z, y = 0$.

$Q =$

(b) If $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ is the rotation by π about the z-axis, find the unit quaternion Q' so that $M_{Q'}(X) = T(M_Q(X))$ for all $X \in \mathbb{R}^3$

$Q' =$

[6] 10. Suppose $R\colon \mathbb{R}^3 \to \mathbb{R}^3$ is reflection in a plane P and that $R(0,0,0) = (1,2,3)$. Find an equation for P:

Answer:

[6] 11. Let the subset S of $\mathbb{R}^4 = \{(x_1, x_2, x_3, x_4)\}$ be defined by the conditions:

$$x_i \geq 0 \text{ for } i = 1,2,3,4 \quad \text{and} \quad x_1 + x_2 + x_3 + x_4 = 1.$$

Then S is a regular 3-dimensional polyhedron. Identify the polyhedron and calculate the length of its edges.

Polyhedron name:

Edge length =