The University of British Columbia

Final Examination - December 13, 2014

MATH 307

Closed book examination. Time: 2.5 hours

Last Name ____________________ **First** ____________________

Signature ____________________ **Student Number** ____________________

Special Instructions:

You are allowed one 3 inch by 5 inch index card of notes (both sides). No memory aids are allowed. No calculators may be used. **Show all your work;** little or no credit will be given for a numerical answer without the correct accompanying work. **Make sure to read the instructions for each question carefully.** If you need more space than the space provided, use the back of the previous page and indicate that you have done so. Where boxes are provided for answers, put your final answers in them.

Rules governing examinations

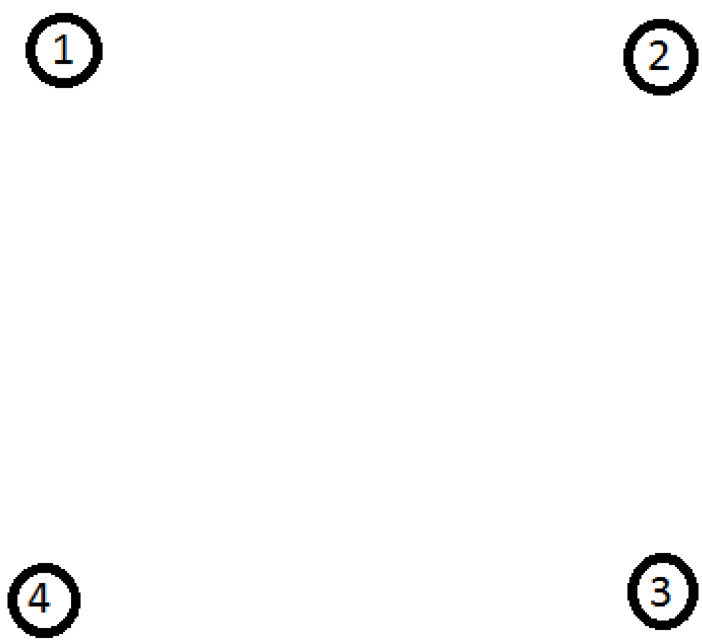
- Each candidate must be prepared to produce, upon request, a UBCcard for identification.
- Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.
- No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination.
- Candidates suspected of any of the following, or similar, dishonest practices shall be immediately dismissed from the examination and shall be liable to disciplinary action.
  (a) Having at the place of writing any books, papers or memoranda, calculators, computers, sound or image players/recorders/transmitters (including telephones), or other memory aid devices, other than those authorized by the examiners.
  (b) Speaking or communicating with other candidates.
  (c) Purposely exposing written papers to the view of other candidates or imaging devices. The plea of accident or forgetfulness shall not be received.
- Candidates must not destroy or mutilate any examination material; must hand in all examination papers; and must not take any examination material from the examination room without permission of the invigilator.
- Candidates must follow any additional examination rules or directions communicated by the instructor or invigilator.

1		13
2		20
3		10
4		10
5		10
6		12
Total		75

1. Consider the following incidence matrix.

$$\begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & -1 \end{bmatrix} \end{array}$$

(a) (3 pts) Beginning with the nodes pictured below, draw the graph corresponding with the above incidence matrix. Recall that rows correspond with edges and columns correspond with nodes. Number each edge as indicated by the number next to each row. Number each node as indicated by the number above each column.



(b) (10 pts) Now suppose that edges 1, 2, and 3 each have resistance equal to one, but edges 4 and 5 have resistance equal to two. What is the effective resistance between nodes 1 and 2? (Put your answer on the next page.)

Answer

2. (10 pts) Consider the differential equation

$$f''(x) + xf(x) = 1, \qquad f(2) = 0, \quad f'(8) = 5$$

in the interval $2 \leq x \leq 8$.

(a) Let $F = [F_0, F_1, F_2, F_3]$ be the finite differences approximation to $f(x)$ when $N = 3$. Write down the matrix equation that F must solve. (Note that the second boundary condition involves the derivative, $f'(8)$.)

Answer

(b) (10 pts) Now suppose $N = 1000$. Below is partial Matlab code to construct the finite differences matrix equation and solve it. Some pieces of code are missing. Fill in the missing pieces. In particular, any time you see blank lines of the form

_________________,

there is a piece of code for you to fill in. (You don't necessarily have to use all of the lines.) Also, **there is one error** somewhere in the code that is already written. Find it and fix it.

```
>>  N = 1000;
>>
>>  % This program sets up and solves a matrix equation of the form
>>  % A F = b, where F is a vector of length N+1 representing the
>>  % discrete approximation to f.
>>
>>  % We first set deltaX to be the length of the interval between
>>  % points in our discretization
>>
>>  deltaX = _____________
>>
>>  % We now construct the matrix L.  We first set it to all zeros
>>  % and then add in entries.
>>  L = zeros(N+1, N+1);
>>
>>  % Now add entries to L corresponding to the boundary conditions.
>>
>>  _______________________________________________________________
>>
>>  _______________________________________________________________
>>
>>  _______________________________________________________________
>>
>>  % Now add the remaining entries to L.
>>
%>>  D = diag([0, -2*ones(1, N - 1), 0]);
%>>  UD = diag([0, ones(1, N)], 1);
%>>  LD = diag([ones(1,N), 0], -1);
%>>  L = D + UD + LD;
>>
>>  % Now we construct Q
>>
>>  _______________________________________________________________
>>
>>  _______________________________________________________________
>>
>>  _______________________________________________________________
>>
```

```
>>  % Now we construct the matrix A.
>>
>>  A = ____________________________________________________________
>>
>>  % Now we construct the vector b
>>
>>  ________________________________________________________________
>>
>>  ________________________________________________________________
>>
>>  ________________________________________________________________
>>
>>  % Now we solve the finite differences matrix equation for F
>>
>>  F = ________________
```

3. (a) (5 pts) Let A be a **3 by 3** matrix of your choosing. From the orthogonality relation, $R(A) = N(A^T)^{\perp}$ we know that it is impossible for $R(A)$ to be equal to $N(A^T)$. However, is there some choice of A which satisfies $R(A) = N(A)$? If so, find one, and give its reduced SVD. If not, why not?

Answer

(b) (5 pts) Let A be a **4 by 4** matrix of you choosing. Is there some choice of A which satisfies $R(A) = N(A)$? If so, find one, and give its reduced SVD. If not, why not?

Answer

4. (10 pts) Let

$$q = \begin{bmatrix} 3 \\ 5 \\ 1 \\ 3 \end{bmatrix}, \quad v = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \quad x = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad y = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \quad z = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

Find a vector, w, which belongs to $\mathrm{span}(v, x, y, z)$, and which, among all such vectors, has minimal Euclidean distance to q. (Part of the challenge is to determine what the question is asking!)

Answer

5. (10 pts) Let A be a matrix with SVD $A = U\Sigma V^*$ (this is the standard SVD, not the reduced SVD). Suppose that

$$\Sigma = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix}.$$

A *right-inverse* of A is a matrix B which satisfies $AB = I$. Does A have two or more right inverses? If so, find two right inverses. If not, why not?

Hint: You may write your answer to the above question in terms of U and V. For example, if the question had been, *what is* A^*A*?* Then the answer would have been

$$A^*A = V\Sigma^*\Sigma V^* = V \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 9 & 0 & 0 \\ 0 & 0 & 0 & 16 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} V^*.$$

Answer

6. Consider the following three decompositions of an n by m, rank-r, matrix A:

$$\text{RREF:} \quad A = LQ$$

where Q is the RREF form of A and L is invertible.

$$\text{SVD:} \quad A = U\Sigma V^*$$

where U and V are unitary and Σ only has non-zero entries on the diagonal.

$$\text{Reduced SVD:} \quad A = \tilde{U}\tilde{\Sigma}\tilde{V}^*$$

where $\tilde{U}$ and $\tilde{V}$ each a have r orthonormal columns, and $\tilde{\Sigma}$ is square diagonal.

For each of the following questions, write down all of the matrices, $L, Q, U, \Sigma, V, \tilde{U}, \tilde{\Sigma}, \tilde{V}$, $U^*, V^*, \tilde{U}^*, \tilde{V}^*$, which necessarily satisfy the requirement. If a matrix does not always satisfy the requirement, do not write it down. For these problems, you do not need to show work, but also there is no partial credit.

(a) (3 pts) Which matrices have the same range as A?

Answer

(b) (3 pts) Which matrices have the same null space as A?

Answer

(c) (3 pts) Which matrices have the same norm as A?

Answer

(d) (3 pts) Which matrices have the same rank as A?

Answer

The End

(If you finish early, please check your work, make sure you carefully read the instructions, make sure you showed work for each problem, and work on the clarity of the presentation of your answers.)