University of British Columbia
Math 307, Final

April 29, 2014
12.00-2.30pm

Name:

Student Number:

Signature:

Instructor:

Instructions:

1. No notes, books or calculators are allowed. A MATLAB/Octave formula sheet is provided.
2. Read the questions carefully and make sure you provide all the information that is asked for in the question.
3. Show all your work. Answers without any explanation or without the correct accompanying work could receive no credit, even if they are correct.
4. Answer the questions in the space provided. Continue on the back of the page if necessary.

Question	Mark	Maximum
1		20
2		13
3		16
4		10
5		14
6		12
7		15
Total		100

1. Consider the following types of matrices (all assumed to be square):

   (A) Matrices with a basis of eigenvectors
   (B) Matrices with distinct eigenvalues
   (C) Matrices with repeated eigenvalues
   (D) Hermitian matrices
   (E) Non-zero orthogonal projection matrices
   (F) Matrices of the form $\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$ with $a \neq 0$.

[2] (a) Which types are always diagonalizable?
(A) □, (B) □, (C) □, (D) □, (E) □, (F) □

Solution:
(A) ☑, (B) ☑, (C) □, (D) ☑, (E) ☑, (F) □

[2] (b) Which types are sometimes, but not always diagonalizable?
(A) □, (B) □, (C) □, (D) □, (E) □, (F) □

Solution:
(A) □, (B) □, (C) ☑, (D) □, (E) □, (F) □

[2] (c) Which types always have an orthonormal basis of eigenvectors?
(A) □, (B) □, (C) □, (D) □, (E) □, (F) □

Solution:
(A) □, (B) □, (C) □, (D) ☑, (E) ☑, (F) □

[2] (d) Which types always have an eigenvalue equal to 1?
(A) □, (B) □, (C) □, (D) □, (E) □, (F) □

Solution:
(A) □, (B) □, (C) □, (D) □, (E) ☑, (F) ☑

[2] (e) Every matrix of type (A) is always also of type:
(A) ☑, (B) □, (C) □, (D) □, (E) □, (F) □

Solution:
(A) ☑, (B) □, (C) □, (D) □, (E) □, (F) □

[2] (f) Every matrix of type (B) is always also of type:
(A) □, (B) ☑, (C) □, (D) □, (E) □, (F) □

Solution:
(A) ☑, (B) ☑, (C) □, (D) □, (E) □, (F) □

[2] (g) Every matrix of type (C) is always also of type:
(A) □, (B) □, (C) ☑, (D) □, (E) □, (F) □

Solution:
(A) □, (B) □, (C) ☑, (D) □, (E) □, (F) □

[2] (h) Every matrix of type (D) is always also of type:
(A) □, (B) □, (C) □, (D) ☑, (E) □, (F) □

Solution:
(A) ☑, (B) □, (C) □, (D) ☑, (E) □, (F) □

[2]

(i) Every matrix of type (E) is always also of type:
(A) ☐, (B) ☐, (C) ☐, (D) ☐, (E) ☑, (F) ☐

Solution:

(A) ☑, (B) ☐, (C) ☐, (D) ☑, (E) ☑, (F) ☐

[2]

(j) Every matrix of type (F) is always also of type:
(A) ☐, (B) ☐, (C) ☐, (D) ☐, (E) ☐, (F) ☑

Solution:

(A) ☐, (B) ☐, (C) ☑, (D) ☐, (E) ☐, (F) ☑

2. We wish to interpolate the points (x_1, y_1), (x_2, y_2) and (x_3, y_3) with $x_1 < x_2 < x_3$ using a function of the form

$$f(x) = \begin{cases} a_1x^2 + b_1x + c_1 & \text{for} \quad x_1 < x < x_2 \\ a_2x^2 + b_2x + c_2 & \text{for} \quad x_2 < x < x_3 \end{cases}$$

[3]

(a) Write down the equations satisfied by $a_1, b_1, c_1, a_2, b_2, c_2$ when $f(x)$ is continuous and passes through the given points.

Solution:

$$\begin{aligned} x_1^2a_1 + x_1b_1 + c_1 &= y_1 \\ x_2^2a_1 + x_2b_1 + c_1 &= y_2 \\ x_2^2a_2 + x_2b_2 + c_2 &= y_2 \\ x_3^2a_2 + x_3b_2 + c_2 &= y_3 \end{aligned}$$

[3]

(b) Write down the equation satisfied by $a_1, b_1, c_1, a_2, b_2, c_2$ when $f'(x)$ is continuous at $x = x_2$.

Solution:

$$2x_2a_1 + b_1 - 2x_2a_2 - b_2 = 0$$

[3]

(c) Write down the matrix A and the vector $\mathbf{b}$ in the matrix equation $A\mathbf{a} = \mathbf{b}$ satisfied by $\mathbf{a} = [a_1, b_1, c_1, a_2, b_2, c_2]^T$ when the conditions of both (a) and (b) are satisfied and when $x_1 = 0, x_2 = 1, x_2 = 2, y_1 = 1, y_2 = 3, y_3 = 2$. Explain why this system of equations does not have a unique solution.

Solution: $A = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 4 & 2 & 1 \\ 2 & 1 & 0 & -2 & -1 & 0 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 3 \\ 3 \\ 2 \\ 0 \end{bmatrix}$. Since A is a 5×6 matrix its null space must be at least one dimensional. This implies any solution will not be unique.

[4] (d) Let A and $\mathbf{b}$ be as in (c) and assume they have been defined in MATLAB/Octave. Using that `a=A\b` computes a solution (even if it is not unique) and `n=null(A)` computes a vector in $N(A)$, write the MATLAB/Octave code that computes and plots two different interpolating functions of the form $f(x)$ satisfying the conditions in (a) and (b).

Solution:

```
a=A\b;
n=null(A);
X1=linspace(0,1,100);
plot(X1,polyval(a(1:3),X1))
hold on
X2=linspace(1,2,100);
plot(X2,polyval(a(4:6),X2))

a1=a+n
X1=linspace(0,1,100);
plot(X1,polyval(a1(1:3),X1))
hold on
X2=linspace(1,2,100);
plot(X2,polyval(a1(4:6),X2))
hold off
```

3. Consider the plane S defined by $2u - 3v + w = 0$, and recall that the normal to this plane is the vector $\mathbf{a} = [2, -3, 1]$.

[3]

(a) Compute the projections of vectors $[1, 0, 0]$ and $[0, 1, 0]$ onto the line spanned by $\mathbf{a}$.

Solution: The projection matrix is $P = \frac{1}{\|\mathbf{a}\|^2}\mathbf{a}\mathbf{a}^T = \frac{1}{14}\begin{bmatrix} 4 & -6 & 2 \\ -6 & 9 & -3 \\ 2 & -3 & 1 \end{bmatrix}$ so the projections are $\mathbf{p}_1 = P\begin{bmatrix}1\\0\\0\end{bmatrix} = \frac{1}{14}\begin{bmatrix}4\\-6\\2\end{bmatrix}$ and $\mathbf{p}_2 = P\begin{bmatrix}0\\1\\0\end{bmatrix} = \frac{1}{14}\begin{bmatrix}-6\\9\\-3\end{bmatrix}$.

[4]

(b) Compute the projections of vectors $[1, 0, 0]$ and $[0, 1, 0]$ onto the subspace defined by S. What is the inner product of each of these projections with $[2, -3, 1]$?

Solution: The complementary projection is $Q = I - P$ so the projections are $\mathbf{q}_1 = (I - P)\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}1\\0\\0\end{bmatrix} - \frac{1}{14}\begin{bmatrix}4\\-6\\2\end{bmatrix} = \frac{1}{14}\begin{bmatrix}10\\6\\-2\end{bmatrix}$ and $\mathbf{q}_2 = (I - P)\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}0\\1\\0\end{bmatrix} - \frac{1}{14}\begin{bmatrix}-6\\9\\-3\end{bmatrix} = \frac{1}{14}\begin{bmatrix}6\\5\\3\end{bmatrix}$ The inner product of each of these projections with $[2, -3, 1]$ is zero.

[3]

(c) Find a basis for the subspace of $\mathbb{R}^3$ defined by S. What is the dimension of this subspace?

Solution: The vectors $\mathbf{q}_1$ and $\mathbf{q}_2$ form a basis. The dimension of this subspace is 2.

[6]

(d) The *reflection* of vector $\mathbf{x}$ across a subspace is $(2P - I)\mathbf{x}$ where I is the identity matrix and P is the matrix projecting $\mathbf{x}$ onto the subspace.

i. Draw a sketch to show why this definition of reflection makes sense.

ii. What is the reflection of $[1, 0, 0]$ in plane S?

iii. What is the matrix $(2P - I)^2$?

Solution:

i. [sketch]

ii. The matrix P in this question is the Q of part (b). $(2Q - I)\begin{bmatrix}1\\0\\0\end{bmatrix} = 2\mathbf{q}_1 - \begin{bmatrix}1\\0\\0\end{bmatrix} = \frac{1}{14}\begin{bmatrix}6\\12\\-4\end{bmatrix}$

iii. $(2P - I)^2 = 4P^2 - 4P + I = 4P - 4P + I = I$ This makes sense because reflecting twice results in the original vector.

4. Consider the following bivariate data:

x	y
-1	0
1	1
3	1

[3]

(a) Draw a sketch showing the approximate least-squares straight-line fit $y = ax + b$ to this data.

Solution: [sketch]

[4]

(b) Write down the least squares (or normal) equation satisfied by $\begin{bmatrix} a \\ b \end{bmatrix}$

Solution: The equation is $A^T A \begin{bmatrix} a \\ b \end{bmatrix} = A^T \mathbf{y}$ where $A = \begin{bmatrix} -1 & 1 \\ 1 & 1 \\ 3 & 1 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$. Explicitly

$$\begin{bmatrix} 11 & 3 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}.$$

[3]

(c) What quantity is minimized by the solution to the equation in (b)?

Solution: The minimized quantity is $\|A \begin{bmatrix} a \\ b \end{bmatrix} - \mathbf{y}\|^2 = (-a+b-0)^2 + (a+b-1)^2 + (3a+b-1)^2$.

[3] 5. Consider $L^2[a,b]$, the set of square-integrable functions on the interval $x \in [a,b]$.

(a) Why do we say that the basis functions $e^{2\pi inx/(b-a)}$ for $n \in \mathbb{Z}$ are orthogonal?

Solution: For $n \neq m$ the inner products

$$\begin{aligned}
\langle e^{2\pi inx/(b-a)}, e^{2\pi imx/(b-a)} \rangle &= \int_a^b e^{2\pi i(m-n)x/(b-a)} dx \\
&= ((b-a)/2\pi i(m-n))(e^{2\pi i(m-n)b/(b-a)} - e^{2\pi i(m-n)a/(b-a)}) \\
&= ((b-a)/2\pi i(m-n))e^{2\pi i(m-n)a/(b-a)}(e^{2\pi i(m-n)(b-a)/(b-a)}) - 1) \\
&= ((b-a)/2\pi i(m-n))e^{2\pi i(m-n)a/(b-a)}(1-1) \\
&= 0.
\end{aligned}$$

[3]

(b) Under what conditions on a and b are these functions orthonormal? Propose a set of basis functions for $L^2[a,b]$ that are orthonormal for any choice of a and b.

Solution: We have

$$\begin{aligned}
\|e^{2\pi inx/(b-a)}\|^2 &= \int_a^b |e^{2\pi inx/(b-a)}|^2 dx \\
&= \int_a^b 1 dx \\
&= b-a.
\end{aligned}$$

so they are normalized if $b-a=1$. The functions $\frac{1}{\sqrt{b-a}} e^{2\pi inx/(b-a)}$ are always orthonormal.

[3] (c) Suppose $a = 0, b = 1$ and consider the function

$$f(x) = \begin{cases} 1, & 0 \leq x < 1/2, \\ -1, & 1/2 \leq x \leq 1 \end{cases}.$$

Write down (but don't bother evaluating) the integral you'd need to do to compute the Fourier coefficients c_n for $f(x)$.

Solution:

$$c_n = \int_0^1 e^{-2\pi inx} f(x)dx = \int_0^{1/2} e^{-2\pi inx}dx - \int_{1/2}^1 e^{-2\pi inx}dx$$

[3] (d) Are the quantities $c_n - c_{-n}$ purely real, purely imaginary, or neither? Why?

Solution:

$$c_n - c_{-n} = \int_0^1 (e^{-2\pi inx} - e^{2\pi inx})f(x)dx = \int_0^1 (-2i)\sin(2\pi nx)f(x)dx$$

is purely imaginary, since $f(x)$ is real.

[2] (e) What is the sum

$$\sum_{n=-\infty}^{\infty} |c_n|^2,$$

where c_n are the Fourier coefficients of the function in part (c)?

Solution: By Parseval's formula this sum is equal to $\int_0^1 |f(x)|^2 dx = int_0^1 dx = 1$.

6. Starting with initial values x_0 and x_1, let x_n for $n = 2, 3, \ldots$ be defined by the recursion relation

$$x_{n+1} = ax_n - x_{n-1},$$

where a is a real number.

[3] (a) When this recursion relation is written in matrix form $X_{n+1} = AX_n$, what are A and X_n?

Solution: $A = \begin{bmatrix} a & -1 \\ 1 & 0 \end{bmatrix}$ and $X_n = \begin{bmatrix} x_{n+1} \\ x_n \end{bmatrix}$

[3] (b) Find the eigenvalues and eigenvectors of A. What is $\det(A)$ and what does it tell you about the eigenvalues?

Solution: The characteristic polynomial is $\lambda^2 - a\lambda + 1 = 0$ so the eigenvalues are

$$\lambda_\pm = \frac{a \pm \sqrt{a^2 - 4}}{2}.$$

The eigenvectors are

$$\mathbf{v}_\pm \begin{bmatrix} \lambda_\pm \\ 1 \end{bmatrix}$$

The determinant is $\det(A) = 1$. this tells us that $\lambda_+\lambda_- = 1$.

[3]

(c) In some applications we are interested in solutions x_n where $\lim_{n\to\infty} x_n = 0$. Find non-zero initial conditions x_0 and x_1 that give rise to such a solution when $a = 3$. **Solution:** When $a = 3$ the eigenvalues are real and positive. Since $\lambda_+\lambda_- = 1$ we know the smaller one λ_- must be less than 1. Thus we choose initial conditions to be the corresponding eigenvector, i.e.,

$$\begin{bmatrix} x_1 \\ x_0 \end{bmatrix} = \begin{bmatrix} (3-\sqrt{5})/2 \\ 1 \end{bmatrix}.$$

[3]

(d) For which values of a do the solutions to this recursion stay bounded, neither growing or decaying as $n \to \infty$? (You may disregard values of a for which A has repeated eigenvalues).

Solution: When the eigenvalues have a non-zero imaginary part then $\lambda_- = \overline{\lambda_+}$ so $|\lambda_+|^2 = |\lambda_-|^2 = 1$. In this case the solutions all stay bounded. This happens when $a^2 < 4$ or $|a| < 2$. When $|a| = 2$ then A has repeated eigenvalues $\lambda_+ = \lambda_- = 1$. We are ignoring this case. When $|a| > 2$ then the eigenvalues are distinct and real. In this case one eigenvalue has absolute value > 1 and the other has absolute value < 1, so there are both growing and decaying solutions.

[3]

7. (a) Write down the definition of a stochastic (or Markov) matrix.

Solution: An $n \times n$ matrix is stochastic if (i) all entries are non-negative and (ii) the entries in each column sum to 1. (equivalently (i) all entries lie in the interval $[0, 1]$ and (ii) the entries in each column sum to 1.)

[5]

(b) What can you say about the relative sizes of $\|S\mathbf{v}\|_1$ and $\|\mathbf{v}\|_1$ for a stochastic matrix S? Explain how this implies that all the eigenvalues λ of a stochastic matrix have $|\lambda| \leq 1$. Is it possible that all eigenvalues have $|\lambda| < 1$? Give a reason. What is $\|S\|_1$ (i.e., the matrix norm when both input and output are measured with the 1-norm, also denoted $\|S\|_{1,1}$)?

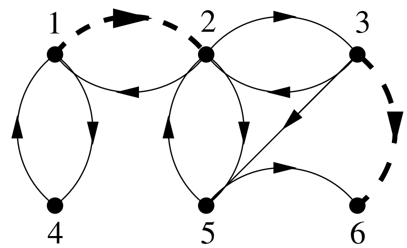
Solution: We know a stochastic matrix S does not increase the 1-norm, that is, for any vector $\mathbf{v}$, $\|S\mathbf{v}\|_1 \leq \|\mathbf{v}\|_1$. If λ is an eigenvalue with eigenvector $\mathbf{v}$ then $S\mathbf{v} = \lambda\mathbf{v}$ and so $\|S\mathbf{v}\|_1 = \|\lambda\mathbf{v}\|_1 = |\lambda|\|\mathbf{v}\|_1$. Thus the inequality implies $|\lambda|\|\mathbf{v}\|_1 \leq \|\mathbf{v}\|_1$. Since $\|\mathbf{v}\|_1 \neq 0$ we can divide to obtain $|\lambda| \leq 1$. Every stochastic matrix has 1 as an eigenvalue, therefore it is not possible that all eigenvalues have $|\lambda| < 1$. The matrix norm $\|S\|_1 = 1$ since the inequality above implies that $\|S\|_1 \leq 1$ and the fact that 1 is an eigenvalue implies that $\|S\|_1 \geq 1$

[3]

(c) What can you say about the eigenvalues of a stochastic matrix S if $\lim_{n\to\infty} S^n$ does not exist. Give an example of a stochastic matrix like this.

Solution: In this case there must be at least two eigenvalues on the unit circle. An example is $S = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ which has eigenvalues 1 and -1.

[4] (d) Consider the following internet.



In this diagram the links depicted by dashed arrows are displayed prominently and are therefore twice as likely to be followed than the remaining links on the page. Write down (i) the stochastic matrix associated to this internet with no damping and (ii) the first column of the stochastic matrix associated to this internet with damping factor 1/2. Explain how you could use the eig command in MATLAB/Octave to compute the limiting probabilies of landing on each site.

Solution: With no damping: $S = \begin{bmatrix} 0 & 1/3 & 0 & 1 & 0 & 1/6 \\ 2/3 & 0 & 1/4 & 0 & 0 & 1/6 \\ 0 & 1/3 & 0 & 0 & 1/2 & 1/6 \\ 1/3 & 0 & 0 & 0 & 0 & 1/6 \\ 0 & 1/3 & 1/4 & 0 & 0 & 1/6 \\ 0 & 0 & 1/2 & 0 & 1/2 & 1/6 \end{bmatrix}$.

With damping factor 1/2 the first column of S is $S = \begin{bmatrix} 1/12 & \ldots \\ 5/12 & \ldots \\ 1/12 & \ldots \\ 3/12 & \ldots \\ 1/12 & \ldots \\ 1/12 & \ldots \end{bmatrix}$.

If `S` is defined in MATLAB/Octave, `[V,D]=eig(S)` computes the eigenvectors and eigenvalues. Assuming that the first diagonal entry of D is the eigenvalue 1, `V(:,1)/sum(V(:,1))` computes the limiting probabilities.

`pi`	π
`i`	$\sqrt{-1}$
`x = 3`	define variable **x** to be 3
`x = [1 2 3]`	set **x** to the 1×3 row vector $(1, 2, 3)$
`x = [1; 2; 3]`	set **x** to the 3×1 vector $(1, 2, 3)$
`A = [1 2; 3 4]`	set A to the 2×2 matrix $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$
`x(2) = 7`	change x_2 to 7
`A(2,1) = 0`	change A_{21} to 0
`3*x`	multiply each element of **x** by 3
`x+3`	add 3 to each element of **x**
`x+y`	add **x** and **y** element by element
`A*x`	product of matrix A and column vector **x**
`A*B`	product of two matrices A and B
`x.*y`	element-wise product of vectors **x** and **y**
`A^3`	for a square matrix A, raise to third power
`cos(A)`	cosine of every element of A
`sin(A)`	sine of every element of A
`x'`	transpose of vector **x**
`A'`	transpose of vector A
`A(2:12,4)`	the submatrix of A consisting of the second to twelfth rows of the fourth column
`A(2:12,4:5)`	the submatrix of A consisting of the second to twelfth rows of the fourth and fifth columns
`A(2:12,:)`	the submatrix of A consisting of the second to twelfth rows of all columns
`A([1:4,6],:)`	the submatrix of A consisting of the first to fourth rows and sixth row
`[A B; C D]`	creates the matrix $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ where A, B, C, D are block matrices (blocks must have compatible sizes)
`rand(12,4)`	12×4 matrix with uniform random numbers in $[0, 1)$
`zeros(12,4)`	12×4 matrix of zeroes
`ones(12,4)`	12×4 matrix of ones
`eye(12)`	12×12 identity matrix
`eye(12,4)`	12×4 matrix whose first 4 rows are the 4×4 identity
`linspace(1.2,4.7,100)`	row vector of 100 equally spaced numbers from 1.2 to 4.7
`diag(x)`	matrix whose diagonal is the entries of **x** (other elements are zero)
`diag(x,n)`	matrix whose diagonal is the entries of **x** on diagonal n (other elements are zero)
`sum(x)`	sum of the elements of **x**

`A\b`	returns the solution **x** to $A\mathbf{x} = \mathbf{b}$
`A^(-1)`	returns the inverse of A
`rref(A)`	returns the reduced row echelon form of A
`det(A)`	returns the determinant of A
`norm(A)`	returns the (operator) norm of A
`cond(A)`	returns the condition number of A
`length(A)`	returns the larger of the number of rows and number of columns of A
`norm(x)`	returns the norm (length) of a vector **x**
`vander(x)`	returns the Vandermonde matrix for the points of **x**
`polyval(a,x)`	returns the values of the polynomial $a_1 x^{n-1} + a_2 x^{n-2} + \ldots a_n$ at the points of **x**
`[Q R] = qr(A,0)`	returns the matrices Q and R in the QR factorization of A
`nextpow2(N)`	calculates the next power of 2 of N
`fft(f,N)`	FFT transform of the vector **f** using N points (pads **f** with zeros if it has fewer than N elements)
`polyval(A)`	returns the coefficients of the characteristic polynomial of A
`roots(a)`	returns the solutions to $a_1 x^{n-1} + a_2 x^{n-2} + \ldots a_n = 0$
`[V D] = eig(A)`	returns the matrix V whose columns are normalized eigenvectors of A and the diagonal matrix D of corresponding eigenvalues
`plot(x,y,'bo')`	plots the points of **y** against the points of **x** using blue dots
`plot(x,y,'r-')`	plots the points of **y** against the points of **x** using red lines
`semilogy(x,y,'bo')`	plots **y** against **x** using a logarithmic scale for **y**
`axis([-0.1 1.1 -3 5])`	changes the axes of the plot to be from -0.1 to 1.1 for the x-axis and -3 to 5 for the y-axis
`hold on`	puts any new plots on top of the existing plot
`hold off`	any new plot commands replace the existing plot (this is the default)
`plot3(x,y,z,'bo')`	plots the points of **z** against the points of **x** and **y** using blue dots
`for k=1:10 ... end`	for loop taking `k` from `1` to `10` and performing the commands ... for each