

The University of British Columbia

Final Examination - April 23, 2013

Mathematics 307/202

Closed book examination

Time: 2.5 hours

Last Name _____ First _____ Signature _____

Student Number _____

Special Instructions:

No books, notes, or calculators are allowed. A MATLAB/Octave formula sheet is provided on the last page.

Rules governing examinations

- Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCcard for identification.
- Candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.
- No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no candidate shall be permitted to enter the examination room once the examination has begun.
- Candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.
- Candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
 - (a) speaking or communicating with other candidates, unless otherwise authorized;
 - (b) purposely exposing written papers to the view of other candidates or imaging devices;
 - (c) purposely viewing the written papers of other candidates;
 - (d) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
 - (e) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s)—(electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).
- Candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.
- Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.
- Candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

1		12
2		15
3		15
4		13
5		15
6		10
7		20
Total		100

[12] 1.

(a) [3 pts] Write down the definition of the matrix norm $\|A\|$ of a matrix A .

(b) [3 pts] Write down the definition of the condition number $\text{cond}(A)$ of a matrix A . Why is this a useful concept?

(c) [3 pts] If A is a 2×2 matrix with $\|A\| = 2$, is it possible that $A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$? Give a reason.

(d) [3 pts] Find the norm and condition number of $\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$ ($= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$).

[15] 2.

- (a) [2 pts] Write down the definition of a Hermitian matrix.
- (b) [3 pts] TRUE or FALSE: Eigenvectors for distinct eigenvalues are orthogonal for a real symmetric matrix. Justify your answer.
- (c) [3 pts] TRUE or FALSE: If a square matrix has repeated eigenvalues, then it cannot be diagonalized. Justify your answer.
- (d) [2 pts] Write down the definition of a stochastic matrix.
- (e) [3 pts] What can you say about the eigenvalues and eigenvectors of a stochastic matrix?
- (f) [2 pts] What can you say about the eigenvalues and eigenvectors of a stochastic matrix if all the entries are strictly positive?

[15] **3.** Suppose that A is a real symmetric matrix.

(a) [5 pts] Explain how to find the largest (in absolute value) eigenvalue of A using the power method.

(b) [5 pts] Explain how to find the eigenvalue of A that is closest to 2 using the power method.

(c) [5 pts] Write down the MATLAB/Octave commands that implement the procedure in (b) with N iterations. Assume that A and N have been defined in MATLAB/Octave, and that the size of A is 1000×1000 .

[13] 4. Let S be the subspace of \mathbb{R}^4 spanned by $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}$. Given the MATLAB/Octave calculation

```
> rref([1 2 4; 1 -1 1;1 -1 1])  
ans =
```

```
1 0 2  
0 1 1  
0 0 0
```

(a) [7 pts] Find the matrix P that projects onto S .

(b) [6 pts] Write down the MATLAB/Octave commands that find the vector in S closest to $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

[15] 5 Suppose A is 3×4 matrix and

$$A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 2 & 2 & 1 & 3 \\ 3 & 3 & 1 & 4 \end{bmatrix} \quad \text{rref}(A) = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) [4 pts] Find a basis for $R(A)$

(b) [4 pts] Find a basis for $N(A)$

(c) [4 pts] Find a basis for $R(A^T)$

(d) [3 pts] What is the rank of A and $\dim(N(A^T))$

[10] **6.**

(a) [5 pts] Find the coefficients c_n in the Fourier series $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n x}$ if

$$f(x) = \begin{cases} 1 & 0 \leq x < 1/2 \\ 0 & 1/2 \leq x \leq 1 \end{cases}$$

(b) [5 pts] Find $\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}$ using Parseval's formula.

[20] 7. We want to interpolate through $(0, 1)$, $(1, 0)$, $(2, 2)$ using cubic splines

$$f(x) = \begin{cases} p_1(x) & 0 \leq x \leq 1 \\ p_2(x) & 1 \leq x \leq 2 \end{cases}$$

(a) [4 pts] Write down $p_1(x)$ and $p_2(x)$ in terms of unknown coefficients.

(b) [4 pts] $f(x)$ must pass through all given points. Write down the linear equations expressing this condition.

(c) [4 pts] Interior derivative must be continuous. Write down the linear equations expressing this condition.

(d) [4 pts] At the endpoints we have zero second derivatives. Write down the linear equations expressing this condition.

(e) [4 pts] Combine equations (a) – (d) into a single matrix equation and write down the MATLAB/Octave commands you need to solve it.

The End

```

pi
1
x = 3
x = [1 2 3]
x = [1; 2; 3]
A = [1 2; 3 4]
x(2) = 7
x(2,1) = 0
3*x
x*x
x+y
A*x
A*B
x.*y
A^-3
cos(A)
sin(A)
x'
A'
A(2:12,4)
A(2:12,4:5)
A(2:12,:)
A([1:4,6],:)
[A B; C D]

rand(12,4)
zeros(12,4)
ones(12,4)
eye(12)
eye(12,4)
linspace(1,2,4,7,100)
diag(x)
diag(x,n)
sum(x)

pi
sqrt(-1)
define variable x to be 3
set x to the 1 x 3 row vector (1,2,3)
set x to the 3 x 1 vector (1,2,3)
set A to the 2 x 2 matrix [ 1 2
                          3 4 ]
change x2 to 7
multiply each element of x by 3
add 3 to each element of x
add x and y element by element
product of matrix A and column vector x
product of two matrices A and B
element-wise product of vectors x and y
for a square matrix A, raise to third power
cosine of every element of A
sine of every element of A
transpose of vector x
transpose of vector A
the submatrix of A consisting of the second to twelfth rows of the fourth column
the submatrix of A consisting of the second to twelfth rows of the fourth and
fifth columns
the submatrix of A consisting of the second to twelfth rows of all columns
the submatrix of A consisting of the first to fourth rows and sixth row
creates the matrix [ A B
                   C D ] where A, B, C, D are block matrices (blocks must
have compatible sizes)
12 x 4 matrix with uniform random numbers in [0,1)
12 x 4 matrix of zeros
12 x 4 matrix of ones
12 x 12 identity matrix
12 x 4 matrix whose first 4 rows are the 4 x 4 identity
row vector of 100 equally spaced numbers from 1.2 to 4.7
matrix whose diagonal is the entries of x (other elements are zero)
matrix whose diagonal is the entries of x on diagonal n (other elements are
zero)
sum of the elements of x

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```

A\b
A^(-1)
rref(A)
det(A)
norm(A)
cond(A)
length(A)
norm(x)
vander(x)
polyval(a,x)
[Q R] = qr(A,0)
nextpow2(N)
fft(f,N)
polyval(A)
roots(a)
[V D] = eig(A)
plot(x,y,'bo')
plot(x,y,'r-')
semilogy(x,y,'bo')
axis([-0.1 1.1 -3 5])
hold on
hold off
plot3(x,y,z,'bo')
for k=1:10 ... end

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returns the solution x to Ax = b
returns the inverse of A
returns the reduced row echelon form of A
returns the determinant of A
returns the (operator) norm of A
returns the condition number of A
returns the larger of the number of rows and number of columns of A
returns the norm (length) of a vector x
returns the Vandermonde matrix for the points of x
returns the values of the polynomial  $a_1x^{n-1} + a_2x^{n-2} + \dots + a_n$  at the points of x
returns the matrices Q and R in the QR factorization of A
calculates the next power of 2 of N
FFT transform of the vector f using N points (pads f with zeros if it has fewer than N elements)
returns the coefficients of the characteristic polynomial of A
returns the solutions to  $a_1x^{n-1} + a_2x^{n-2} + \dots + a_n = 0$ 
returns the matrix V whose columns are normalized eigenvectors of A and the diagonal matrix D of corresponding eigenvalues
plots the points of y against the points of x using blue dots
plots the points of y against the points of x using red lines
plots y against x using a logarithmic scale for y
changes the axes of the plot to be from -0.1 to 1.1 for the x-axis and -3 to 5 for the y-axis
puts any new plots on top of the existing plot
any new plot commands replace the existing plot (this is the default)
plots the points of z against the points of x and y using blue dots
for loop taking k from 1 to 10 and performing the commands ... for each

```