The University of British Columbia

Final Examination - April 23, 2013

Mathematics 307/202

Closed book examination — Time: 2.5 hours

Last Name ____________________ **First** _____________ **Signature** __________________

Student Number ____________________

Special Instructions:

No books, notes, or calculators are allowed. A MATLAB/Octave formula sheet is provided on the last page.

1		12
2		15
3		15
4		13
5		15
6		10
7		20
Total		100

[12] **1.**

(a) [3 pts] Write down the definition of the matrix norm $\|A\|$ of a matrix A.

(b) [3 pts] Write down the definition of the condition number $\text{cond}(A)$ of a matrix A. Why is this a useful concept?

(c) [3 pts] If A is a 2×2 matrix with $\|A\| = 2$, is it possible that $A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$? Give a reason.

(d) [3 pts] Find the norm and condition number of $\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} \left(= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \right)$.

[15] **2**.

(a) [2 pts] Write down the definition of a Hermitian matrix.

(b) [3 pts] TRUE or FALSE: Eigenvectors for distinct eigenvalues are orthogonal for a real symmetric matrix. Justify your answer.

(c) [3 pts] TRUE or FALSE: If a square matrix has repeated eigenvalues, then it cannot be diagonalized. Justify your answer.

(d) [2 pts] Write down the definition of a stochastic matrix.

(e) [3 pts] What can you say about the eigenvalues and eigenvectors of a stochastic matrix?

(f) [2 pts] What can you say about the eigenvalues and eigenvectors of a stochastic matrix if all the entries are strictly positive?

[15] **3**. Suppose that A is a real symmetric matrix.

(a) [5 pts] Explain how to find the largest (in absolute value) eigenvalue of A using the power method.

(b) [5 pts] Explain how to find the eigenvalue of A that is closest to 2 using the power method.

(c) [5 pts] Write down the MATLAB/Octave commands that implement the procedure in (b) with N iterations. Assume that A and N have been defined in MATLAB/Octave, and that the size of A is 1000×1000.

[13] **4**. Let S be the subspace of $\mathbb{R}^4$ spanned by $\begin{bmatrix}1\\1\\1\end{bmatrix}$, $\begin{bmatrix}2\\-1\\-1\end{bmatrix}$ and $\begin{bmatrix}4\\1\\1\end{bmatrix}$. Given the MATLAB/Octave calculation

```
> rref([1 2 4; 1 -1 1;1 -1 1])
ans =

   1   0   2
   0   1   1
   0   0   0
```

(a) [7 pts] Find the matrix P that projects onto S.

(b) [6 pts] Write down the MATLAB/Octave commands that find the vector in S closest to $\begin{bmatrix}1\\0\\0\end{bmatrix}$

[15] **5** Suppose A is 3×4 matrix and

$$A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 2 & 2 & 1 & 3 \\ 3 & 3 & 1 & 4 \end{bmatrix} \quad \text{rref}(A) = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) [4 pts] Find a basis for $R(A)$

(b) [4 pts] Find a basis for $N(A)$

(c) [4 pts] Find a basis for $R(A^T)$

(d) [3 pts] What is the rank of A and $\dim(N(A^T))$

[10] **6**.

(a) [5 pts] Find the coefficients c_n in the Fourier series $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n x}$ if

$$f(x) = \begin{cases} 1 & 0 \le x < 1/2 \\ 0 & 1/2 \le x \le 1 \end{cases}$$

(b) [5 pts] Find $\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}$ using Parseval's formula.

[20] **7**. We want to interpolate through $(0, 1)$, $(1, 0)$, $(2, 2)$ using cubic splines

$$f(x) = \begin{cases} p_1(x) & 0 \le x \le 1 \\ p_2(x) & 1 \le x \le 2 \end{cases}$$

(a) [4 pts] Write down $p_1(x)$ and $p_2(x)$ in terms of unknown coefficients.

(b) [4 pts] $f(x)$ must pass through all given points. Write down the linear equations expressing this condition.

(c) [4 pts] Interior derivative must be continuous. Write down the linear equations expressing this condition.

(d) [4 pts] At the endpoints we have zero second derivatives. Write down the linear equations expressing this condition.

(e) [4 pts] Combine equations $(a) - (d)$ into a single matrix equation and write down the MATLAB/Octave commands you need to solve it.

The End

`pi`	π
`i`	$\sqrt{-1}$
`x = 3`	define variable $\mathbf{x}$ to be 3
`x = [1 2 3]`	set $\mathbf{x}$ to the 1×3 row vector $(1, 2, 3)$
`x = [1; 2; 3]`	set $\mathbf{x}$ to the 3×1 vector $(1, 2, 3)$
`A = [1 2; 3 4]`	set A to the 2×2 matrix $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$
`x(2) = 7`	change x_2 to 7
`A(2,1) = 0`	change A_{21} to 0
`3*x`	multiply each element of $\mathbf{x}$ by 3
`x+3`	add 3 to each element of $\mathbf{x}$
`x+y`	add $\mathbf{x}$ and $\mathbf{y}$ element by element
`A*x`	product of matrix A and column vector $\mathbf{x}$
`A*B`	product of two matrices A and B
`x.*y`	element-wise product of vectors $\mathbf{x}$ and $\mathbf{y}$
`A^3`	for a square matrix A, raise to third power
`cos(A)`	cosine of every element of A
`sin(A)`	sine of every element of A
`x'`	transpose of vector $\mathbf{x}$
`A'`	transpose of vector A
`A(2:12,4)`	the submatrix of A consisting of the second to twelfth rows of the fourth column
`A(2:12,4:5)`	the submatrix of A consisting of the second to twelfth rows of the fourth and fifth columns
`A(2:12,:)`	the submatrix of A consisting of the second to twelfth rows of all columns
`A([1:4,6],:)`	the submatrix of A consisting of the first to fourth rows and sixth row
`[A B; C D]`	creates the matrix $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ where A, B, C, D are block matrices (blocks must have compatible sizes)
`rand(12,4)`	12×4 matrix with uniform random numbers in $[0, 1)$
`zeros(12,4)`	12×4 matrix of zeroes
`ones(12,4)`	12×4 matrix of ones
`eye(12)`	12×12 identity matrix
`eye(12,4)`	12×4 matrix whose first 4 rows are the 4×4 identity
`linspace(1.2,4.7,100)`	row vector of 100 equally spaced numbers from 1.2 to 4.7
`diag(x)`	matrix whose diagonal is the entries of $\mathbf{x}$ (other elements are zero)
`diag(x,n)`	matrix whose diagonal is the entries of $\mathbf{x}$ on diagonal n (other elements are zero)
`sum(x)`	sum of the elements of $\mathbf{x}$

`A\b`	returns the solution $\mathbf{x}$ to $A\mathbf{x} = \mathbf{b}$
`A^(-1)`	returns the inverse of A
`rref(A)`	returns the reduced row echelon form of A
`det(A)`	returns the determinant of A
`norm(A)`	returns the (operator) norm of A
`cond(A)`	returns the condition number of A
`length(A)`	returns the larger of the number of rows and number of columns of A
`norm(x)`	returns the norm (length) of a vector $\mathbf{x}$
`vander(x)`	returns the Vandermonde matrix for the points of $\mathbf{x}$
`polyval(a,x)`	returns the values of the polynomial $a_1x^{n-1} + a_2x^{n-2} + \dots a_n$ at the points of $\mathbf{x}$
`[Q R] = qr(A,0)`	returns the matrices Q and R in the QR factorization of A
`nextpow2(N)`	calculates the next power of 2 of N
`fft(f,N)`	FFT transform of the vector $\mathbf{f}$ using N points (pads $\mathbf{f}$ with zeros if it has fewer than N elements)
`polyval(A)`	returns the coefficients of the characteristic polynomial of A
`roots(a)`	returns the solutions to $a_1x^{n-1} + a_2x^{n-2} + \dots a_n = 0$
`[V D] = eig(A)`	returns the matrix V whose columns are normalized eigenvectors of A and the diagonal matrix D of corresponding eigenvalues
`plot(x,y,'bo')`	plots the points of $\mathbf{y}$ against the points of $\mathbf{x}$ using blue dots
`plot(x,y,'r-')`	plots the points of $\mathbf{y}$ against the points of $\mathbf{x}$ using red lines
`semilogy(x,y,'bo')`	plots $\mathbf{y}$ against $\mathbf{x}$ using a logarithmic scale for $\mathbf{y}$
`axis([-0.1 1.1 -3 5])`	changes the axes of the plot to be from -0.1 to 1.1 for the x-axis and -3 to 5 for the y-axis
`hold on`	puts any new plots on top of the existing plot
`hold off`	any new plot commands replace the existing plot (this is the default)
`plot3(x,y,z,'bo')`	plots the points of $\mathbf{z}$ against the points of $\mathbf{x}$ and $\mathbf{y}$ using blue dots
`for k=1:10 ... end`	for loop taking `k` from `1` to `10` and performing the commands ... for each