Marks

[15] **1.** Let

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 2 \end{bmatrix},$$

where a is a real number.

(a) [4] For what values of a (if any) does the matrix norm have the value $\|A\| = 2$?

(b) [2] For what values of a (if any) is $\mathrm{cond}(A)$ not defined? Give a reason.

(c) [2] For what values of a (if any) is $\mathrm{cond}(A) = 1/2$? Give a reason.

Continued on page 3

(d) [4] Sketch a graph of $\text{cond}(A)$ as a function of a for $-\infty < a < \infty$.

(e) [3] For what values of a (if any) is $\text{cond}(A) = 4$?

Continued on page 4

[15] **2.** Suppose that

$$A = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 2 & 2 & 1 & 3 & 0 \\ 3 & 3 & 1 & 4 & 1 \\ 4 & 4 & 1 & 5 & 1 \end{bmatrix} \quad \mathrm{rref}(A) = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) [3] Write down a basis for $R(A)$

(b) [3] Write down a basis for $N(A)$

(c) [3] Write down a basis for $R(A^T)$

(d) [3] What are rank(A) and dim($N(A^T)$)?

(e) [3] Write down the MATLAB/Octave commands that would compute the projection of $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ onto $R(A)$.

Continued on page 5

[12] **3.** Suppose we are given 4 points (x_1, y_1), (x_2, y_2), (x_3, y_3) and (x_4, y_4) in the plane and we want to find a function $f(x)$, defined for $x_1 \le x \le x_4$, whose graph interpolates these points. Assume that

$$f(x) = \begin{cases} p_1(x) & \text{for } x_1 \le x \le x_2 \\ p_2(x) & \text{for } x_2 \le x \le x_3 \\ p_3(x) & \text{for } x_3 \le x \le x_4 \end{cases}$$

where each $p_i(x)$ is a polynomial.

(a) [3] What equations, written in terms of $p_i(x)$ and possibly their derivatives, express the condition that $f(x)$ goes through the given points? Do these equations imply that $f(x)$ is continuous?

(b) [3] What equations, written in terms of $p_i(x)$ and possibly their derivatives, express the condition that $f'(x)$ is continuous?

(c) [3] What equations, written in terms of $p_i(x)$ and possibly their derivatives, express the condition that $f''(x)$ is continuous?

(d) [3] When each $p_i(x)$ is a cubic polynomial of the form $a_i(x-x_i)^3+b_i(x-x_i)^2+c_i(x-x_i)+d_i$ the equations written in parts (a), (b) and (c) above are equivalent to a system of linear equations in the unknowns a_i, b_i, c_i and d_i, $i = 1, 2, 3$. How many more equations are needed if there are to be the same number of equations as unknowns? What equations are usually added and why?

Continued on page 6

[10] **4.** In this question we are once again given 4 points (x_1, y_1), (x_2, y_2), (x_3, y_3) and (x_4, y_4) in the plane. This time we want to find a quadratic function $q(x) = ax^2 + bx + c$ that comes closest to going through the points by doing a least squares fit.

(a) [5] The least squares equation you need to solve to find the coeficients a, b and c has the form $A^T A\mathbf{a} = A^T\mathbf{b}$. Write down expressions for A, $\mathbf{a}$, and $\mathbf{b}$.

(b) [5] Suppose the points (x_i, y_i) have been defined in MATLAB/Octave as `X1, ..., X4`, `Y1, ... Y4`. Write down the MATLAB/Octave code that plots these points, then computes $q(x)$, and finally plots $q(x)$.

Continued on page 7

[15] **5.** Define a sequence x_0, x_1, $\ldots$ by the initial conditions $x_0 = a$, $x_1 = b$ and $x_2 = c$ together with the recursion relation

$$x_{n+3} = x_{n+2} + x_{n+1} + x_n$$

for $n = 0, 1, 2, \ldots$.

(a) [7] Rewrite this recursion in matrix form $X_{n+1} = AX_n$ for $n = 0, 1, 2, \ldots$ for a sequence X_n of vectors, with an initial vector X_0 and some matrix A.

Continued on page 8

(b) [5] If the matrix `A` from part (a) is defined in MATLAB/Octave, we can do the following calculations:

```
>eig(A)
ans =

   1.83929 + 0.00000i
  -0.41964 + 0.60629i
  -0.41964 - 0.60629i
```

```
> abs(eig(A))
ans =

   1.83929
   0.73735
   0.73735
```

Describe how you could make further use of the `eig` command and other MATLAB/Octave commands to determine all (possibly complex) initial values a, b and c for which $x_n \to 0$ as $n \to \infty$.

(c) [3] Explain how you could ensure that the a, b and c you find in part (b) are real numbers.

Continued on page 9

[18] **6.**

(a) [3] Determine the coefficients c_n in the expansion $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n x}$, where $f(x) = x$ and $0 \le x \le 1$.

(b) [3] Calculate the inner product $\langle f(x), f(x) \rangle$ for $f(x) = x$ on the interval $0 \le x \le 1$, using the definition of the inner product for functions.

Continued on page 10

(c) [3] Explain how the orthogonality of the functions $e^{2\pi i n x}$ allows you to relate the inner product in part (b) to the sum $\sum_{n=-\infty}^{\infty} |c_n|^2$. Use your answer to to calculate the infinite sum $\sum_{n=1}^{\infty} n^{-2}$.

(d) [3] What points in the plane would you plot to produce a frequency-amplitude plot for the function in part (a)?

Continued on page 11

(e) [3] Explain how you could use the `fft` command in MATLAB/Octave to compute approximations to the coefficients c_n in part (a). Write down the commands you would use, and say for what values of n you would expect your approximations to be most accurate.

(f) [3] Suppose you expanded the same function $f(x) = x$ as in part (a) except on the interval $0 \le x \le 2$. What would be the *form* (i.e., do not compute the coefficients) of the Fourier series valid for this interval. What points on the plane would you plot to produce a frequency-amplitude plot from this new Fourier series? (Give the answer in terms of the coefficients in the new expansion.)

Continued on page 12

[15] **7.** Suppose A is a symmetric 4×4 matrix with eigenvalues $0, 1, 4, 5$. Define a sequence of vectors $\mathbf{x}_n \in \mathbb{R}^4$ by choosing $\mathbf{x}_0$ at random, and then setting

$$\mathbf{y}_n = (A - 3I)^{-1}\mathbf{x}_{n-1}$$
$$\mathbf{x}_n = \mathbf{y}_n / \|\mathbf{y}_n\|$$

for $n = 1, 2, \ldots$. You then observe that $\mathbf{x}_n$ converges to $\mathbf{x}_\infty = [1/2, 1/2, 1/2, 1/2]^T$ as $n \to \infty$.

(a) [0] What is $A\mathbf{x}_\infty$?

(b) [0] What is the value of the inner (dot) product $\langle \mathbf{x}_\infty, A\mathbf{x}_\infty \rangle$?

(c) [0] What vector does $\mathbf{y}_n$ converge to?

The End

Be sure that this examination has 12 pages including this cover

The University of British Columbia
Sessional Examinations - December 2010

Mathematics 307: Applied Linear Algebra
Section 101 - Richard Froese.

Closed book examination | Time: 2.5 hours

Print Name ____________________ **Signature** ____________________

Student Number ____________________ **Instructor's Name** ____________________

Section Number ____________________

Special Instructions:

No calculators, cell phones, formula sheets, or books are allowed.
A list of useful MATLAB/Octave commands is provided on the final page of this booklet.

Rules governing examinations

1. Each candidate should be prepared to produce his or her library/AMS card upon request.
2. Read and observe the following rules:

No candidate shall be permitted to enter the examination room after the expiration of one half hour, or to leave during the first half hour of the examination.

Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.

CAUTION - Candidates guilty of any of the following or similar practices shall be immediately dismissed from the examination and shall be liable to disciplinary action.

(a) Making use of any books, papers or memoranda, other than those authorized by the examiners.

(b) Speaking or communicating with other candidates.

(c) Purposely exposing written papers to the view of other candidates. The plea of accident or forgetfulness shall not be received.

3. Smoking is not permitted during examinations.

1		15
2		15
3		12
4		10
5		15
6		18
7		15
Total		100