University of British Columbia: Math 307

Final examination

Date: April 25, 2009
Time: 8:30 am

Name (print):
Student ID Number:
Signature:

Instructions:

1. No notes or books are allowed. You will be provided with a sheet of MATLAB/Octave commands.
2. No calculators are permitted.
3. Read the questions carefully and make sure you provide all the information that is asked for in the question.
4. Show all your work. Answers without any explanation or without the correct accompanying work could receive no credit, even if they are correct.
5. Answer the questions in the space provided. Continue on the back of the page if necessary.

This examination consists of 16 pages (including this cover sheet). Check to ensure that it is complete.

Rules governing formal examinations

1. Each candidate must be prepared to produce, upon request, a UBCcard for identification;
2. Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions;
3. No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination;
4. Candidates suspected of any of the following, or similar, dishonest practices shall be immediately dismissed from the examination and shall be liable to disciplinary action:
   - Having at the place of writing any books, papers or memoranda, calculators, computers, sound or image players/recorders/transmitters (including telephones), or other memory aid devices, other than those authorized by the examiners;
   - Speaking or communicating with other candidates;
   - Purposely exposing written papers to the view of other candidates or imaging devices. The plea of accident or forgetfulness shall not be received.
5. Candidates must not destroy or mutilate any examination material; must hand in all examination papers; and must not take any examination material from the examination room without permission of the invigilator.
6. Candidates must follow any additional examination rules or directions communicated by the instructor or invigilator.

Qu.	Mark	Maximum
1		14
2		14
3		14
5		14
6		14
7		15
8		15
9		15
Total		100

[14] 1. (a) Calculate the 1- and 2-norms of the following vectors:

$$\mathbf{a} = (2, -3, 1) \qquad \text{and} \qquad \mathbf{b} = (2 + 3i, 2 - 3i).$$

(b) Define the matrix norm for a matrix A.

[Continued on next page]

(c) Given the matrix

$$A = \begin{bmatrix} -10 & 0 & 0 \\ 0 & -3.5 & 0 \\ 0 & 0 & 0.5 \end{bmatrix}$$

Determine the matrix norm of A and $\mathrm{cond}(A)$.

(d) In trying to solve the equation $A\mathbf{x} = \mathbf{b}$, where A is the matrix in part (c), you are able to measure $\mathbf{b}$ with an accuracy of $\|\Delta\mathbf{b}\|/\|\mathbf{b}\| \leq 0.1$. What is the largest possible relative error $\|\Delta\mathbf{x}\|/\|\mathbf{x}\|$ in your solution for $\mathbf{x}$?

[14] 2. (a) A cubic spline $f(x)$ interpolating three points (x_1, y_1), (x_2, y_2) and (x_3, y_3) can be written as

$$f(x) = \begin{cases} p_1(x) & \text{if } x_1 \leq x \leq x_2 \\ p_2(x) & \text{if } x_2 \leq x \leq x_3 \end{cases}$$

where $p_1(x)$ and $p_2(x)$ are polynomials. What conditions do these polynomials have to satisfy?

(b) Write down a matrix equation that you would need to solve in order to find the cubic spline passing through the points $(1, 2)$, $(2, 4)$, and $(3, 3)$.

[14] 3. Let

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}.$$

(a) Write down a 4×3 matrix A such that the equation $A\mathbf{c} = \mathbf{0}$ has a non-zero solution $\mathbf{c}$ if and only if $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ are linearly dependent.

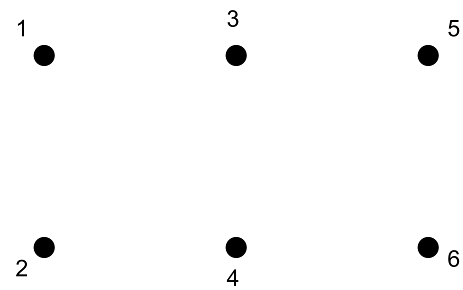
(b) Explain how you could use MATLAB/Octave to determine whether $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ are linearly dependent. What output would tell you that the vectors are independent?

[Continued on next page]

(c) Given that $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ are in fact linearly independent, what MATLAB/Octave commands compute the projection of $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ onto the span of $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$?

(d) What MATLAB/Octave commands compute the projection of $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ onto the direction orthogonal to $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$?

[14] 4. (a) Draw and label the connections between the nodes in the diagram



so that the incidence matrix of the resulting graph is given by

$$D = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{bmatrix}$$

(b) Suppose the resulting graph represents a resistor network with all resistances equal to 1. Write down down the Laplacian matrix L.

[**Continued on next page**]

(c) If the entries of the vector $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{bmatrix}$ represent the voltages at the nodes of the graph, what do the entries of $L\mathbf{v}$ represent?

(d) A battery is attached to nodes 1 and 2 with voltages $v_1 = 0$ and $v_2 = 1$. Write down the MATLAB/Octave commands that compute the voltages $v_3, \ldots, v_6$ for the rest of the network. Assume that the matrix L has already been defined in MATLAB/Octave.

[14] 5. The function $e^{i\pi x}$ for $x \in [0,1]$ can be expanded in a Fourier series

$$e^{i\pi x} = \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n x}$$

(a) Calculate c_0 and c_1. Simplify as much as possible. ($\int e^{iax}\,dx = \frac{e^{iax}}{ia} + C$ for any $a \neq 0$.)

[Continued on next page]

(b) Suppose $\phi_0(x), \phi_1(x), \phi_2(x), \ldots$ is an infinite orthonormal basis of functions on the interval $[0, 1]$ and we have the expansion

$$e^{i\pi x} = \sum_{n=0}^{\infty} d_n \phi_n(x).$$

Suppose, in addition, that ϕ_0 is the constant function $\phi_0(x) = 1$. Is it true that $d_0 = c_0$, where c_0 is the Fourier coefficient above? Give a reason.

For questions 6, 7 & 8 choose and answer any TWO of the questions. Note that only two of the questions will be marked. Indicate which two you would like to have marked by circling the appropriate question numbers.

[15] 6. (a) Define the algebraic multiplicity and geometric multiplicity for an eigenvector λ. Under what conditions is a matrix diagonalizable?

(b) Given a non-singular matrix A, and an "initial guess" vector $\mathbf{x}$, write out the steps involved in the algorithm for the power method for calculating an eigenvector corresponding to the *smallest* (in magnitude) eigenvalue.

[Continued on next page]

(c) Suppose

$$A = \begin{bmatrix} -11 & 10.5 & -4.5 \\ -9 & 8.5 & -4.5 \\ 12 & -12 & 4 \end{bmatrix}$$

After many iterations of the steps in part (b) you find $\mathbf{x}_{100} = \begin{bmatrix} 0.7071 \\ 0.7071 \\ 0.0000 \end{bmatrix}$, $\mathbf{x}_{101} = \begin{bmatrix} -0.7071 \\ -0.7071 \\ 0.0000 \end{bmatrix}$, and $A\mathbf{x}_{100} = \begin{bmatrix} -0.3536 \\ -0.3536 \\ 0.0000 \end{bmatrix}$. Determine the corresponding eigenvalue eigenvector pair.

(d) Describe a case when the power method may fail to converge.

[15] 7. Given the recurrence relation

$$x_{n+1} = 3x_n - 2x_{n-1}$$

with the initial condition $x_0 = a$ and $x_1 = b$,

(a) Solve the recurrence relation. (Give a general *scalar* expression for x_n in terms of n, a, and b).

[Continued on next page]

(b) For what values of a and b will the sequence $x_0, x_1, x_2, \ldots$ converge to a finite limit?

[15] 8. A player begins a game of chance by placing a marker in box 2, marked *start.* A die is rolled, and the marker is moved one square to the left if a 1 or 2 is rolled and one square to the right if a 3, 4, 5, or 6 is rolled. This process continues until the marker lands in square 1, in which case the player wins the game, or in square 4, in which case the player loses the game.

1	2	3	4
win	start		lose

(a) Write a stochastic matrix that represents the behaviour of the game. Label the columns with the corresponding "states" of the game. (*Hint:* Once you win, you win forever; and once you lose, you lose forever; i.e., if you reach state 1 at some point you will continue to stay at state 1 forever-after.)

(b) How would you check in MATLAB/Octave whether the above stochastic matrix has only one eigenvalue equal to 1 or multiple eigenvalues equal to 1? Write down the commands you would use and explain how the output would indicate which of the possibilities is true.

[Continued on next page]

(c) Suppose you are told that the eigenvalues of the stochastic matrix are

$$\lambda_1 = 1, \qquad \lambda_2 = 1, \qquad \lambda_3 = \frac{\sqrt{2}}{3}, \qquad \lambda_4 = -\frac{\sqrt{2}}{3},$$

and the corresponding eigenvalues are

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \qquad \mathbf{v}_3, \qquad \mathbf{v}_4,$$

and that the eigenvectors form a basis of $\mathbb{R}^4$. Write down a matrix equation that you would solve in order to find the unique set of coefficients $\{c_1, c_2, c_3, c_4\}$ to express the initial state of the game, $\mathbf{x}_0 = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}^T$, in terms of the above basis of eigenvectors. Just write the equation symbolically using $\mathbf{v}_i$, c_i, and $\mathbf{x}_0$.

(d) Given that when the initial state $\mathbf{x}_0$ is written in terms of the eigenvectors, $\mathbf{v}_i$, the coefficients $(c_1, c_2, c_3, c_4) = (0.43, 0.57, 0.28, 0.93)$, determine the probability of winning the game. Explain your solution by writing a expression for the long-time behaviour of the game in terms of c_i, λ_i, and $\mathbf{v}_i$.