Math 307 Final Exam
Dec 3, 2008
Duration: 150 minutes

Last Name: ________________ First Name: ____________ Student Number: ________

Do not open this test until instructed to do so! This exam should have 12 pages, including this cover sheet. No textbooks, calculators, or other aids are allowed. One page of notes is allowed. Turn off any call phones, pagers, etc. that could make noise during the exam. You must remain in this room until you have finished the exam. **Circle your solutions! Reduce your answer as much as possible. Explain your work.** Relax. Use the back of the page if necessary.

Read these UBC rules governing examinations:

(i) Each candidate must be prepared to produce, upon request, a Library/AMS card for identification.

(ii) Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.

(iii) No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination.

(iv) Candidates suspected of any of the following, or similar, dishonest practices shall be immediately dismissed from the examination and shall be liable to disciplinary action.

- Having at the place of writing any books, papers or memoranda, calculators, computers, audio or video cassette players or other memory aid devices, other than those authorized by the examiners.
- Speaking or communicating with other candidates.
- Purposely exposing written papers to the view of other candidates. The plea of accident or forgetfulness shall not be received.

(v) Candidates must not destroy or mutilate any examination material; must hand in all examination papers; and must not take any examination material from the examination room without permission of the invigilator.

Problem	**Out of**	**Score**
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
Total	100	

Problem 1 (10 points)

Let

$$A = \begin{pmatrix} 2 & 4 & -6 & 0 & -4 \\ -1 & -2 & 3 & 0 & 2 \\ 3 & 6 & -9 & 0 & -6 \end{pmatrix}.$$

Determine the rank of A and find a basis of the left null space $N(A^\top)$ of A.

Problem 2 (10 points)

Let $\mathbb{P}_3$ be the vector space of polynomials of degree ≤ 3. The monomials $1, x, x^2, x^3$ form a basis of $\mathbb{P}_3$. Consider the linear mapping from $\mathbb{P}_3$ into $\mathbb{P}_3$ given by $p(x) \mapsto p''(x)$. Find the matrix A of the mapping with respect to the monomial basis and determine its null space.

Problem 3 (10 points)

For the parameter α, consider the linear system:

$$\begin{array}{rcrcrcr} x & - & y & + & z & = & 2, \\ -2x & + & y & + & \alpha z & = & -3, \\ x & + & \alpha y & - & z & = & 1. \end{array}$$

Determine when the system has a unique solution, no solution, or infinitely many solutions. (Don't determine the actual solutions!)

Problem 4 (10 points)

Find the QR decomposition of the matrix

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & -1 \\ -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Problem 5 (10 points)

Consider the matrix

$$A = \begin{pmatrix} 0 & 2 & 3 & -1 \\ -2 & 8 & 3 & 1 \\ 0 & -1 & -4 & 3 \\ 0 & 2 & 3 & 1 \end{pmatrix}.$$

Compute $\det(A)$ and $\det(A^3)$.

Problem 6 (10 points)

Let u be a unit vector and $Q = I - 2uu^\top$. Show:

(a) The matrix Q is symmetric and orthogonal.

(b) The matrix $Q + iI$ is invertible.

Problem 7 (10 points)
Consider the symmetric matrix

$$A = \begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{pmatrix}.$$

Compute the matrix norms $\|A\|_\infty$ and $\|A\|_2$.

Problem 8 (10 points)

Consider the linear differential equation

$$\begin{aligned} x'(t) &= 3x(t) + 4y(t), \\ y'(t) &= 3x(t) + 2y(t), \end{aligned}$$

with the initial conditions $x(0) = 6$, $y(0) = 1$. Find the solution $u(t) = (x(t), y(t))^\top$ and determine the stability of the differential equation.

Problem 9 (10 points)

Find the singular values of

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then determine the diagonal matrix Σ in the singular value decomposition $A = U\Sigma V^\top$.

Problem 10 (10 points)

Let $A^H = -A$ be a skew-Hermitian complex matrix. Show that the matrix e^{At} is unitary.

Extra Page