Math 307 Final Exam (April 2006)

Last Name: ______________________________ **First name**: ______________________________

Student #: ______________________________ **Signature**: ______________________________

Circle your section #: **201 (Peterson), 202 (Li)**

I have read and understood the instructions below:

Please sign:

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Instructions:

1. No notes or books are to be used in this exam. No calculators are allowed.
2. Justify every answer, and show your work. Unsupported answers will receive no credit.
3. You will be given 2.5 hrs to write this exam. Read over the exam before you begin. You are asked to stay in your seat during the last 5 minutes of the exam, until all exams are collected.
4. At the end of the exam you will be given the instruction "Put away all writing implements and remain seated." *Continuing to write after this instruction will be considered as cheating.*
5. **Academic dishonesty:** Exposing your paper to another student, copying material from another student, or representing your work as that of another student constitutes academic dishonesty. Cases of academic dishonesty may lead to a zero grade in the exam, a zero grade in the course, and other measures, such as suspension from this university.

Question	grade	value
1		12
2		12
3		12
4		16
5		20
6		12
7		16
Total		**100**

Question 1: **[12 marks]**

Let $PA = LU$ be $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 & 3 \\ 2 & 1 & 8 & 8 \\ 0 & 1 & 4 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 4 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, and $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$.

(a) Find a basis for each of the four fundamental subspaces.

(b) Find the condition on b_1, b_2, and b_3 so that $Ax = b$ has at least one solution.

Question 2: **[12 marks]**

Consider the diagonalizable matrix $A = \begin{bmatrix} 1 & 0 \\ 1 & 0.5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}^{-1}$.

(a) Find the solution $x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$ of the difference equation $x(t+1) = Ax(t)$ satisfying the initial condition $x(0) = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$. Determine the limit of the ratio $x_1(t)/x_2(t)$ as $t \longrightarrow \infty$.

(b) Find the solution $x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$ of the differential equation $\dfrac{dx(t)}{dt} = Ax(t)$ satisfying the initial condition $x(0) = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$.

(*Question 2 continued!*)

Question 3: **[12 marks]**

Let $P_2 = \{\alpha_0 + \alpha_1 t + \alpha_2 t^2 : \ \alpha_j \ \in \ \mathbb{R}\ (j = 0, 1, 2)\}$ be the set of all polynomials of degree at most 2.

(a) Show that P_2 is a vector space and show that $\mathcal{B} = \{1,\ t,\ t^2\}$ form a basis of P_2. Express the function $f(t) = 1 - 3t + 2t^2$ ($f(t) \ \in \ P_2$) in a vector form using $\mathcal{B}$ as the basis.

(b) Show that the transformation $T : \ P_2 \ \rightarrow \ P_2$ is a linear transformation, where $T(f(t)) = (t+1)\dfrac{df(t)}{dt}$ (e.g. $T(1 - 3t + 2t^2) = (t + 1)(-3 + 4t) = -3 + t + 4t^2$).

(c) Find the transformation matrix A such that $T(f) = Af$ using $\mathcal{B}$ as the basis for P_2.

(*Question 3 continued!*)

Question 4: **[16 marks]**

The following discrete dynamical system describes the yearly migration of wild horse populations among three areas R, G, and B. Let $r(t)$, $g(t)$, and $b(t)$ be the sizes of the horse population in areas R, G, and B respectively at the t^{th} year.

$$\vec{x}(t+1) = \begin{bmatrix} r(t+1) \\ g(t+1) \\ b(t+1) \end{bmatrix} = \begin{bmatrix} r(t)/2 + g(t)/3 + b(t)/3 \\ r(t)/2 + g(t)/3 + b(t)/2 \\ g(t)/3 + b(t)/6 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/3 & 1/3 \\ 1/2 & 1/3 & 1/2 \\ 0 & 1/3 & 1/6 \end{bmatrix} \begin{bmatrix} r(t) \\ g(t) \\ b(t) \end{bmatrix} = A\vec{x}(t),$$

where the Markov matrix A describes how the horses move among these areas from one year to the next. The 1st column indicates that each year 1/2 of the horses in area R remain in area R and 1/2 will migrate to area G. The 2nd column shows that horses in area G will be evenly distributed in the three areas one year later. The 3rd column implies that, of the horses in area B, 1/3 will migrate to area R, 1/2 will migrate to area G, and only 1/6 will remain in area B.

We assume that no horses are lost and no new horses are added and that initially (i.e. at $t = 0$), there are a total of 350 horses all located in area B. Thus, $\vec{x}(0) = [0 \;\; 0 \;\; 350]^T$.

(a) Show that $\lambda_3 = 1$ is an eigenvalue of A. Then, find the other two eigenvalues λ_1 and λ_2 .

(b) Find a vector $\vec{v}$ such that $A\vec{v} = \vec{v}$, and thus $A^t\vec{v} = \vec{v}$ for all $t > 1$.

(c) Find a matrix S so that $S^{-1}AS = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$, where λ_1, λ_2, λ_3 are the eigenvalues found in (a).

(d) Find $\vec{x}(t)$ in terms of the eigenvalues and eigenvectors of A. Then, calculate $\lim_{t\to\infty} \vec{x}(t)$.

(*Question 4 continued!*)

(*Question 4 continued!*)

Question 5: **[20 marks]**

A matrix A and a vector b are given by $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$, $\qquad b = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$.

(a) Find an orthonormal basis for the column space of A (i.e. for $\mathcal{R}(A)$). Express the matrix A in the form $A = QR$, where Q is a matrix with orthonormal columns and R is upper triangular and with positive diagonal entries.

(b) Find an orthonormal basis for $\mathbb{R}^3$: $\{u_1,\ u_2,\ u_3\}$ in which $u_1,\ u_2$ span the column space $\mathcal{R}(A)$ of the matrix A. How does the third basis vector u_3 relate to the fundamental subspaces of A?

(c) Split b into $b = b^{\|} + b^{\perp}$, where $b^{\|}$ is in $\mathcal{R}(A)$ and $b^{\perp}$ is in $\mathcal{R}(A)^{\perp}$ (i.e. the orthogonal complement of $\mathcal{R}(A)$.)

(d) Determine if $Ax = b$ is solvable. Find x if it is solvable and find $\bar{x}$ (i.e. the least squares solution) if it is not.

(*Question 5 continued!*)

Question 6: **[12 marks]**

Determine if each of the following statements is true or false. Show *reason or proof* if true and show *reason* or *counter example* if false.

(a) If AB is defined, then $\mathcal{R}(AB) \subset \mathcal{R}(A)$ and $rank(AB) \leq rank(A)$ (i.e. the column space of AB is contained in the column space of A and the rank of AB is at most equal to that of A.)

(b) If $Ax = b$ and $A^T y = 0$, then $b \perp y$ (You cannot use the fundamental theorem. This is actually asking you to prove part of the theorem.)

(*Question 6 continued!*)

(c) Let $U,\ V,\ W$ be three subspaces of $\mathbb{R}^n$. If $U \perp V$ and $V \perp W$, then $U \perp W$.

(d) If an $n \times n$ matrix A is real-valued and if $Av = \lambda v$, $A^T w = \mu w$, then $v \perp w$ if $\lambda \neq \mu$. (In other words, every eigenvector of A is orthogonal to every eigenvector of A^T if they correspond to different eigenvalues.)

Question 7: **[16 marks]**

Let P be the projection of all vectors in $\mathbb{R}^4$ into $\mathcal{R}(A)$ which is the column space of the matrix $A = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 1 & 1 \\ 0 & -2 \end{bmatrix}$. Let R be the reflection in $\mathcal{R}(A)$. Answer the following questions with little or no calculation. You do not need to find the projection and the reflection matrices to answer these questions.

(a) Find all eigenvalues of the projection P and the reflection R.

(b) Find a complete set of eigenvectors for both P and R.

(*Question 7 continued!*)

(c) For the projection P, determine the relation between its row space and its column space and the relation between its nullspace and left nullspace. Then, find a basis for each one of the four fundamental subspaces.

(d) Are P and R invertible? For each invertible transformation among these, find the inverse transformation.