Be sure that this examination has 2 pages.

The University of British Columbia

Final Exam Examinations - December 2012

Mathematics 305

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Closed book examination. No notes, texts, or calculators allowed. Time: $2\frac{1}{2}$ hours

Marks

[30] **1.** Identify whether each of the following statements are true or false. You must give reasons for your answers to receive credit. (Hint: very little calculation is needed to solve these).

(i) $\text{Log}(z_1 z_2) = \text{Log}(z_1) + \text{Log}(z_2)$ for any complex numbers $z_1 \neq 0$ and $z_2 \neq 0$. Here $\text{Log}(z)$ denotes the principal branch of $\log z$.

(ii) Let z and w be any two complex numbers. Then, $|zw| = |z||w|$.

(iii) Let n be a positive integer. Then $\cos(n\theta) = \text{Re}\left[(\cos\theta + i\sin\theta)^n\right]$.

(iv) Let $u(x,y) = e^{-nx}\cos(ny)$ where n is an integer. Then, $u(x,y)$ is a harmonic function.

(v) Suppose that $u(x,y)$ is a harmonic function that is bounded, i.e. that satisfies $u \leq M$ for all x and y, where M is a constant. Then, u must be the constant function.

(vi) Let $p(z) = 6z^3 + z^2 + z + 1$. Then, $p(z)$ has all of its roots in the unit disk $|z| < 1$. (Hint: consider the polynomial $q(w) = w^3 p(1/w)$.)

[15] **2.** Let $f(z) = (z^3 - 1)^{1/2}$. We seek to construct a branch of $f(z)$ that is analytic in $|z| < 1$, and that satisfies $f(0) = i$.

(i) Show how to construct this branch by choosing an appropriate branch of the multi-valued logarithm.

(ii) For the branch that you constructed in (i), determine explicitly where the branch cuts are in the complex plane.

(iii) For your branch of $f(z)$ in (i) calculate $\lim_{y\to 0^-} f(2+iy)$.

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[15] **3.** Calculate each of the following integrals over the simple closed curve C:

(i) $I = \int_C \frac{e^{iz}}{z(z-\pi)}\, dz$ where C is the counter-clockwise circle $|z| = 4$. Can you evaluate this integral by deforming C to a large circle $|z| = R$ and letting $R \to \infty$?

(iii) $I = \int_C \frac{1}{z^3(z+6)}\, dz$ where C is the counter-clockwise circle $|z| = 2$.

(iv) $I = \int_C \frac{1}{z(1-z)} dz$ where C is the counter-clockwise circle $|z| = r$ with $0 < r < 1$. Show also that a direct parameterization of the integral leads to the identity that

$$\int_0^{2\pi} \left(\frac{1 - r\cos\theta}{r^2 + 1 - 2r\cos\theta} \right) d\theta = 2\pi\,,$$

for any r in $0 < r < 1$.

[10] **4.** Consider the polynomial $p(z) = z^4 + z^3 + 5z^2 + 2z + 4$.

(i) Prove that this polynomial has no roots in the right half-plane $\mathrm{Re}(z) > 0$.

(ii) Show that any solution $y(t)$ to the differential equation $y'''' + y''' + 5y'' + 2y' + 4y = 0$ must tend to zero as $t \to +\infty$. (Here the derivatives are with respect to t).

[20] **5.** Calculate the following integrals in as simplified a form as you can:

$$\text{(i)} \quad I = \int_{-\infty}^{\infty} \frac{\cos x}{(x^2 + a^2)^2}\, dx\,, \quad a > 0 \text{ real}\,,$$

$$\text{(ii)} \quad I = \int_0^{\infty} \frac{x^{m-1}}{1 + x^n}\, dx\,, \quad 0 < m < n\,, \quad \text{with } m, n\,, \quad \text{both integers}\,,$$

$$\text{(iii)} \quad I = \int_0^{2\pi} \frac{1}{a + \cos^2\theta}\, d\theta\,, \quad a > 1 \text{ real}\,.$$

(Hint: For (ii) take an appropriate wedge-shaped contour).

[10] **6.** For any positive integer N, let C_N denote the boundary (in the counterclockwise direction) of the rectangle with vertices at

$$(N + \frac{1}{2})(1 + i)\,,\ (N + \frac{1}{2})(-1 + i)\,,\ (N + \frac{1}{2})(-1 - i)\,,\ (N + \frac{1}{2})(1 - i)\,.$$

Define I_N by

$$I_N = \int_{C_N} \frac{\pi}{z^2 \sin(\pi z)}\, dz\,.$$

(i) Draw the rectangle. Determine and then classify the singularities of the integrand.

(ii) Prove directly that $I_N \to 0$ as $N \to \infty$. (Hint: show that $|1/\sin(\pi z)|$ is bounded on C_N, and then use the standard estimate for integrals.)

(iii) By using the residue theorem, together with the results in part (i) and (ii), establish the identity

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} = -\frac{\pi^2}{12}\,.$$

[100] **Total Marks**

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