Be sure that this examination has 2 pages.

The University of British Columbia

Final Examinations - December 2010

Mathematics 305

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Closed book examination. No notes, texts, or calculators allowed. Time: $2\frac{1}{2}$ hours

Special Instructions: No notes, book, or calculator allowed

Marks

[40] **1.** Identify whether each of the following statements are true or false. You must give reasons for your answers.

(i) $\text{Arg}(z_1 z_2) = \text{Arg}(z_1) + \text{Arg}(z_2)$.

(ii) $\text{Re}\,(i/\bar{z}) = -\text{Im}(z)/|z|^2$.

(iii) $\sin(n\theta) = \text{Im}\{(\cos\theta + i\sin\theta)^n\}$ where n is a positive integer.

(iv) $f(z) = |z|^2$ is analytic at $z = 0$ but not at any other point.

(v) $u = r^n\cos(n\theta)$ is a harmonic function, where n is a positive integer, $r^2 = x^2 + y^2$ and $\tan\theta = y/x$.

(vi) If $f(z) = u + iv$ is an entire function, then $u^2 - v^2$ is a harmonic function.

(vii) Let $M = \max(|e^{iz^2}|)$ over the disk $|z| \leq 2$. Then, $M = 1$.

(viii) $|\sin(z)|$ is bounded as $|z| \to \infty$.

(ix) the equation $\sqrt{z} + (1 - i) = 0$, where $\sqrt{z}$ is the principal branch of the square root function, has no solution.

(x) $|e^{z^2}| \leq e^{|z|^2}$ for all z.

(xi) $\log(e^z) = z$.

(xii) $\int_C z^{-1/2}\sin(\sqrt{z})dz = 0$ where C is the simple closed curve $|z| = 1$ oriented counterclockwise, and $\sqrt{z}$ is the principal branch of the square root function.

[15] **2.** Consider the function $f(z)$ defined by

$$f(z) = \frac{z}{z^2 - z - 2},$$

Continued on page 2

(i) Determine the Laurent series of $f(z)$ centered at $z_0 = 0$ that converges in the region $|z| > 2$.

(ii) By using the Laurent series in (i), and by integrating it term by term, evaluate $\int_C f(z)\,dz$ where C is the simple closed curve $|z| = 4$ oriented counterclockwise. Confirm your result by using the residue theorem applied to the function $f(z)$ on the region $|z| \leq 4$.

[15] **3.** Consider the following function $f(z)$ defined by

$$f(z) = \frac{1}{z\,(1-\cos(\sqrt{z}))\,(z-\pi^2)}\,.$$

(i) Identify and then classify all of the singular points of $f(z)$ in the complex plane.

(ii) Calculate $\int_C f(z)dz$ where C is the circle $|z| = 10$ oriented in a counterclockwise sense.

[15] **4.** Let $a > 0$ with a real. By using residue theory, calculate values for the following integrals in as compact a form as you can:

$$\text{(i)} \quad I = \int_0^{2\pi} \frac{1}{a+\cos\theta}\,d\theta\,, \quad \text{with } a > 1; \qquad \text{(ii)} \quad I = \int_0^{\infty} \frac{x\sin x}{x^2+a^2}\,dx\,.$$

[15] **5.** By using residuee theory, calculate the following integrals:

$$\text{(i)} \quad I = \int_0^{\infty} \frac{\sin x}{x(x^2+1)}\,dx\,; \qquad \text{(ii)} \quad I = \int_0^{\infty} \frac{\sqrt{x}}{(x^2+1)}\,dx\,.$$

[100] **Total Marks**

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