Be sure this exam has 11 pages including the cover

The University of British Columbia

Sessional Exams – 2015 Term 2
Mathematics 303 Introduction to Stochastic Processes
Dr. G. Slade

Last Name: ______________________ **First Name:** ______________________

Student Number: ____

This exam consists of **8** questions worth **10** marks each. No aids are permitted. Please show all work and calculations. Numerical answers need not be simplified.

Problem	total possible	score
1.	10	
2.	10	
3.	10	
4.	10	
5.	10	
6.	10	
7.	10	
8.	10	
total	80	

1. Consider the Markov chain with state space $\{0, 1, 2\}$ and transition matrix

$$\mathbf{P} = \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 0.8 & 0.2 & 0 \\ 1 & 0.1 & 0.8 & 0.1 \\ 2 & 0 & 0.2 & 0.8 \end{array}$$

(4 marks) (a) Suppose $X_0 = 0$. Find the probability that $X_2 = 0$.

(4 marks) (b) Find the stationary distribution of the Markov chain.

(2 marks) (c) Suppose $X_0 = 0$. What is the expected number of steps until the first time the Markov chain will return to state 0?

2. Smith gambles constantly at four different casinos, and when he leaves one casino he goes to another which is chosen from the other three with equal probabilities $\frac{1}{3}$. He owns just one umbrella, which he can store at the casinos. Each trip between casinos, he takes his umbrella if it is raining and if he has it at his starting casino, and he never takes the umbrella if it is not raining. It rains independently each trip with probability p.

(4 marks) (a) Let X_n be the number of umbrellas at his current location at the start of the n^{th} trip. This defines a Markov Chain. Write down its transition matrix.

(4 marks) (b) Determine the stationary distribution of the Markov Chain.

(2 marks) (c) In the long run, what fraction of trips does Smith get wet?

3. At all times, and urn contains 2 balls—white balls and/or black balls. At each step, a coin having probability p of landing heads is flipped (with $0 < p < 1$). If heads appears, then a ball is chosen at random from the urn and is replaced by a white ball; if tails appears, then a ball is chosen at random from the urn and is replaced by a black ball. Let X_n denote the number of white balls in the urn after the n^{th} step. This defines a Markov chain.

(4 marks) (a) Calculate the one-step transition matrix for this Markov chain.

(4 marks) (b)) Using any method, determine the stationary distribution for the chain.

(2 marks) (c) In the long run, what fraction of time does the urn contain two white balls?

4. Suppose that X, Y are exponential random variables with parameters λ, μ, respectively, and let U be a uniform random variable on $[0, 1]$.

(2 marks) (a) Fix $s > 0$. Show that $P(X > 2s) = P(X > s)^2$.

(4 marks) (b) Determine whether or not it is true that $P(X > 2U) = P(X > U)^2$.

(4 marks) (c) Find $P(X > x|X < Y)$.

5. Customers arrive according to a Poisson process of rate 30 per hour. Each customer is served and leaves immediately upon arrival. There are two kinds of service, and a customer pays \$5 for Service A or \$15 for Service B. Customers independently select Service A with probability $\frac{1}{3}$ and Service B with probability $\frac{2}{3}$.

(2 marks) (a) What is the distribution of the number of customers who select Service A during the period 9:00am to 9:10am? (Give its name and any parameter(s).)

(3 marks) (b) During the period 9:30–10:30am, there were 32 customers in total. What is the probability that none of them arrived during 10:25–10:30am?

(2 marks) (c) What is the probability that the first two customers after 9:00am request Service B?

(3 marks) (d) Determine the expected amount paid by all customers during a 10 minute period.

(10 marks) 6. Smith has a small booth where he sells lottery tickets. Customers arrive according to a Poisson process of rate λ per minute. He will close the shop on the first occasion that a minutes have elapsed since the last customer arrived. Let X be the amount of time that he keeps the shop open. Show that $EX = \lambda^{-1}(e^{\lambda a} - 1)$ minutes.
Hint: condition on the arrival time of the first customer.

7. Customers arrive, at rate λ, at a business with a single server. The service time has rate μ. Customers will wait in line if there is at most one other customer already waiting (in addition to one being served), but otherwise will go away. This defines a birth and death process with state space $\{0, 1, 2, 3\}$.

(2 marks) (a) What are the birth and death rates?

(6 marks) (b) Determine the limiting probabilities.

(2 marks) (c) Suppose $\lambda = 2$ and $\mu = 1$. What fraction of potential customers are lost, and what fraction of the time is the server idle?

8. A site on the surface of a bacterium is such that foreign molecules, some acceptable and some not, become attached. The foreign molecules arrive at the site according to a Poisson process of rate $\lambda = 3$. Among these molecules, a proportion $\alpha = \frac{1}{3}$ are acceptable. An unacceptable molecule remain at the site for a length of time that is exponentially distributed with rate $\mu_1 = 4$, whereas an acceptable molecule remains at the site for an exponential time of rate $\mu_2 = 2$. An arriving molecule becomes attached only if the site is not already occupied by another molecule. Consider this as a continuous time Markov Chain with states:

   0: no molecule is attached,

   1: an unacceptable molecule is attached,

   2: an acceptable molecule is attached.

(4 marks) (a) Write down the balance equations for the limiting probabilities.

(4 marks) (b) Solve the balance equations to obtain the limiting probabilities.

(2 marks) (c) What proportion of time is the site occupied by an acceptable molecule?