Math 302 Final

Instructor: Asaf Nachmias, section 201 Duration: 2.5 hours

Instructions:

- Write your name and student ID on **every** page.
- This examination contains six questions with weight 17 points each (102 points total).
- Write each answer **very clearly** below the corresponding question (Use back of page if needed). Simplify your answer as much as possible but answers may include factorials, "choose" symbols or the exponential function. You may also use the function $\Phi(a) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{-x^2/2} dx$ in your answer.
- Put a box around each of your final computations.
- No calculators, books, notebooks or any other written materials are allowed.
- **Good luck!**

1. (a) Carefully define (with formulas) what it means for three events A, B and C, to be independent.

   (b) Let A, B, C be independent events with $P(A) = P(B) = P(C) = 1/2$.

      i. Compute $P(A \cup B \cup C)$.

      ii. Let X be the indicator r.v. of the event $A \cup B$ and Y the indicator r.v. of the event $B \cup C$ (that is, $X = 1$ if $A \cup B$ occurred and 0 otherwise, $Y = 1$ if $B \cup C$ occurred and 0 otherwise). Compute $\mathrm{E}[XY]$.

2. (a) Die #1 has 6 sides numbered $1, \ldots, 6$ and die #2 has 8 sides numbered $1, \ldots, 8$. One of these two dice is chosen at random and rolled 10 times. Find the conditional probability that you have selected die #1 given that precisely three 1's were rolled.

(b) Let X and Y be independent Poisson random variables with mean 1. Are $X - Y$ and $X + Y$ independent? Justify your answer.

(c) Let X be a geometric random variable with parameter p. Find $E[\min(X, 5)]$.

3. Five distinct families arrive to a party. Each family consists of 3 people. The 15 participants of the party are arranged randomly in a line.

   (a) What is the probability that the members of the Smith family sit next to each other?

   (b) What is the probability that all the members of the Smith family sit next to each other, but not all the members of the Johnson family sit next to each other?

   (c) Let X be the number of families that their members sit next to each other. Find $\mathrm{E}[X]$ and $\mathrm{Var}(X)$.

4. Let R be the triangle in the $x - y$ plane with corners at $(-1, 0), (0, 1)$ and $(1, 0)$. Assume (X, Y) is uniformly distributed over R, that is, X and Y have a joint density which is a constant c on R, and equal to 0 on the complement of R.

   (a) Find c.

   (b) Find the marginal densities of X and Y.

   (c) Are X and Y independent? Justify your answer.

   (d) Are X and Y uncorrelated? Justify your answer.

5. (a) Let Z be a normal random variable with mean 0 and variance 3. Compute $\mathrm{E}\big[|Z|\big]$.

(b) Let X and Y be independent exponential random variables with mean 1. Find the density function of $X - Y$.

6. The waiting time in hours of Mrs. Cohen at the clinic is a continuous random variable with density

$$f(x) = \begin{cases} cy(2-y) & \text{if } 0 \leq y \leq 2 \\ 0 & \text{else} \end{cases} .$$

(a) Find c.

(b) What is the probability that she waits more than an hour?

(c) Mrs. Cohen goes to the clinic each day for 100 days. The waiting time in each day is independent and has the same distribution. Let A be the event that Mrs. Cohen waits more than an hour in at least ($\geq$) 60 days. Use Markov's inequality to bound $\mathrm{P}(A)$.

(d) Use the central limit theorem to approximate $\mathrm{P}(A)$.