The University of British Columbia

Math 302 — Introduction to Probability

2012, December 10

Instructor: Omer Angel

Surname: _____ First Name: _____ Student id. _____

Instructions

• **Explain** your reasoning thoroughly, and **justify** all answers (even if the question does not specifically say so). Put a box around your final answer.
• Calculators, but no other aides are permitted.
• If you need more space, additional paper is available. Always **note** which questions are solved in a separate booklet, or they may not be graded.
• Duration: **150** minutes.

Good luck, and enjoy the break.

Rules governing examinations

• Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCcard for identification.

• Candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.

• No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no candidate shall be permitted to enter the examination room once the examination has begun.

• Candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.

• Candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
(a) speaking or communicating with other candidates, unless otherwise authorized;
(b) purposely exposing written papers to the view of other candidates or imaging devices;
(c) purposely viewing the written papers of other candidates;
(d) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
(e) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s)–(electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).

• Candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.

• Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.

• Candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

Page	Points	Score
1	15	
2	10	
3	10	
4	20	
5	15	
6	10	
7	10	
8	15	
Total:	105	

Table 1: Common Distributions

Distribution	p.m.f. / p.d.f.	Mean	Variance
$\text{Bin}(n,p)$	$\binom{n}{k}p^k(1-p)^{n-k}$	np	$np(1-p)$
$\text{Geom}(p)$	$p(1-p)^k$	$1/p$	$\dfrac{1-p}{p^2}$
$\text{Poisson}(\lambda)$	$e^{-\lambda}\dfrac{\lambda^n}{n!}$	λ	λ
$\text{Uniform}(a,b)$	$\dfrac{1}{b-a}$	$\dfrac{a+b}{2}$	$\dfrac{(b-a)^2}{12}$
$\text{Exp}(\lambda)$	$\lambda e^{-\lambda x}$	$1/\lambda$	$1/\lambda^2$
$\text{N}(\mu,\sigma^2)$	$\dfrac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$	μ	σ^2

The normal CDF:

x	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990

15 marks 1. (a) Define precisely the covariance of random variables X and Y.

(b) Define precisely the correlation coefficient of random variables X and Y.

(c) Define precisely what it means for events A, B, C to be independent.

10 marks 2. A fair coin is tossed 4 times.

(a) What is the probability of getting exactly 3 heads?

(b) What is the probability of getting exactly 3 heads conditioned on the event that the first two tosses came out the same?

5 marks 3. Let A, B be events so that $\mathbb{P}(A) = 0.5$, $\mathbb{P}(B) = 0.4$ and $\mathbb{P}(A \cup B) = 0.7$. What is $\mathbb{P}(A|B)$?

5 marks 4. If $X = \text{Exp}(1)$ and $Y = \text{Bin}(n, p)$ are independent, what is $\mathbb{P}(X > Y)$?

20 marks

5. Consider variables (X, Y) which are uniformly distributed with density a over the triangle with corners $(0, 0)$, $(6, 0)$ and $(6, 3)$.

   (a) Find a

   (b) find the marginal densities of X and Y

   (c) Find $\mathbb{E}XY$.

   (d) Find $\mathbb{P}(X > 6Y)$.

15 marks 6. (a) Precisely state the central limit theorem.

(b) Suppose the weight of a person has mean 75 (Kg) and variance $\sigma^2 = 100$. An airline has 400 passengers on a flight. Assume their weights are independent, and use the CLT to estimate the probability that their total weight exceeds 30500.

(c) Use Chebyshev's inequality to give a bound on the probability that the total weight exceeds 30500.

10 marks 7. If Z_1, Z_2 are independent $N(0,1)$ random variables, what is the distribution of each of the following:

(a) $2Z_1 + Z_2$

(b) $2Z_1 - Z_2$

10 marks 8. (a) Let $X = \text{Poi}(\lambda)$ for some $\lambda > 0$. For which values of t is $\mathbb{E}e^{tX}$ finite? When it is finite, what is $\mathbb{E}e^{tX}$?

(b) Repeat the same for $Y = \text{Exp}(\lambda)$.

15 marks

9. Alice and Bob arrange the digits $1 \ldots 9$ in independent random orders, and compare the resulting numbers digit by digit. Let Q be the number of digits in agreement. For example, if the numbers happen to be 475619283 and 374956182, then $Q = 2$ (the 7 and 8 are in the same position).

   (a) what approximation rule gives an estimate for the distribution of Q?

   (b) Find $\mathbb{E}Q$ (exactly!)

   (c) Find $\mathrm{Var}(Q)$ (exactly!)