University of British Columbia
Math 301, Section 201

Final exam

Date: April 20, 2013
Time: 8:30 - 11:00am

Name (print):
Student ID Number:
Signature:

Instructor: Richard Froese

Instructions:

1. No notes, books or calculators are allowed. A summary sheet with properties of Fourier and Laplace transforms is provided.
2. Read the questions carefully and make sure you provide all the information that is asked for in the question.
3. Show all your work. Answers without any explanation or without the correct accompanying work could receive no credit, even if they are correct.
4. Answer the questions in the space provided. Continue on the back of the page if necessary.

Question	Mark	Maximum
1		14
2		15
3		15
4		14
5		14
6		14
7		14
Total		100

[14] 1. Evaluate

$$f(x) = \int_{-\infty}^{\infty} \frac{e^{ikx}}{(k^2+1)^2} dk.$$

[5] 2. (a) What are the branch points of the multivalued function $f(z) = (z^2 + 1)^{1/2}$? Is infinity a branch point?

[5] (b) Using the range of angles method, define a branch of $f(z)$ that is continous and positive for $z \in \mathbb{R}$.

[5] (c) Using the range of angles method, define a branch of $f(z)$ that is analytic outside the unit circle and negative for large positive $z \in \mathbb{R}$.

3. In this question we evaluate

$$I = \int_0^\infty \frac{\sqrt{x}}{x^2+1}\,dx.$$

[3]

(a) Draw a diagram that shows the contour you are using, and the branch cut for $\sqrt{z}$.

[4]

(b) What does the Cauchy residue theorem say for your contour? Compute the residues that appear.

[4]

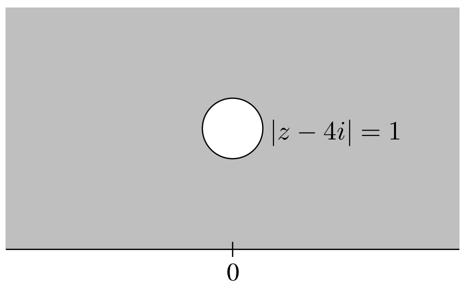
(c) Which integrals (over portions of your contour) tend to zero in the limit? Give estimates that show this.

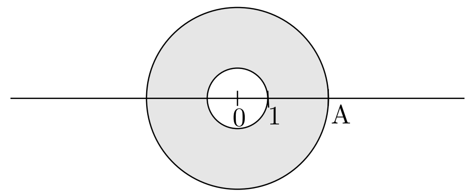
[4]

(d) Evaluate I.

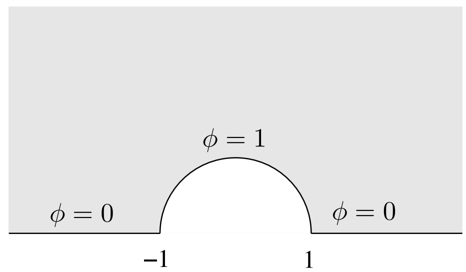
[14]

4. Find a fractional linear transformation that maps the shaded region $\{z : \mathrm{Im} z \geq 0, |z - 4i| \geq 1\}$ to the annulus $\{z : 1 \leq |z| \leq A\}$ with the real line mapping to the unit circle. What is the outer radius A? (Recall that α and α^* are symmetric with respect to a circle centred at a with radius R if $\alpha^* = a + R^2/(\overline{\alpha} - \overline{a})$.)





5. Consider the following region with boundary data.



[7]

(a) Show that the Joukowski map $J(z) = (1/2)(z + 1/z)$ maps the shaded region onto the upper half plane.

[7]

(b) Solve Laplaces equation $\Delta\phi = 0$ in the shaded region, with the indicated boundary conditions.

[7] 6. (a) What is the Fourier transform of $x^2 e^{-x^2/2}$?

[7] (b) Use the Fourier transform to compute $\int_{-\infty}^{\infty} e^{-|x-y|} e^{-|y|} dy$. (Hint: you can use the result from an earlier problem for the last step.)

[7] 7. (a) Compute the Laplace transform $Y(s)$ of the solution $y(t)$ to

$$y''''(t) + y'(t) - y(t) = \cos(t),$$

$$y'''(0) = 1, \quad y''(0) = y'(0) = y(0) = 0,$$

[7] (b) Use the Nyquist criterion to decide whether the solution $y(t)$ grows exponentially as $t \to \infty$ or not.

Fourier Transform Summary

	$f(x)$	$\widehat{f}(k)$
Definition and inversion		
	$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\widehat{f}(k)dk$	$\int_{-\infty}^{\infty} e^{-ikx}f(x)dx$
Examples		
1	$\dfrac{1}{1+x^2}$	$\pi e^{-\lvert k\rvert}$
2	$e^{-\lvert x\rvert}$	$\dfrac{2}{1+k^2}$
3	$\begin{cases} 1 & \lvert x\rvert < 1 \\ 1/2 & x = 0 \\ 0 & \lvert x\rvert \geq 1 \end{cases}$	$2\,\dfrac{\sin(k)}{k}$
4	$\dfrac{\sin(x)}{x}$	$\pi \begin{cases} 1 & \lvert k\rvert < 1 \\ 1/2 & k = 0 \\ 0 & \lvert k\rvert > 1 \end{cases}$
5	$e^{-x^2/(2\sigma^2)}$	$\sqrt{2\pi}\,\sigma e^{-\sigma^2 k^2/2}$
Properties		
0	$c_1 f_1(x) + c_2 f_2(x)$	$c_1\widehat{f}_1(k) + c_2\widehat{f}_2(k)$
1	$f(x+a)$	$e^{iak}\widehat{f}(k)$
2	$e^{iax}f(x)$	$\widehat{f}(k-a)$
3	$f(ax)$	$a^{-1}\widehat{f}(k/a)$
4	$f'(x)$	$ik\widehat{f}(k)$
5	$ixf(x)$	$-\widehat{f}'(k)$
6	$f * g(x)$	$\widehat{f}(k)\widehat{g}(k)$
7	$f(x)g(x)$	$(2\pi)^{-1}\widehat{f} * \widehat{g}(k)$

Note about the examples 3 and 4: The value of a Fourier or inverse Fourier transform is insensitive to changes of the input function at a single point. In property 6: $f * g(x) = \int_{-\infty}^{\infty} f(x-y)g(y)dy$

Laplace Transform Summary

	$y(t)$	$Y(s)$
Definition and inversion		
	$\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{st}Y(s)ds$	$\int_0^{\infty} e^{-st}y(t)dt$
Examples		
1	e^{-at}	$\dfrac{1}{s+a}$
2	$\sin(\omega t)$	$\dfrac{\omega}{s^2+\omega^2}$
3	$\cos(\omega t)$	$\dfrac{s}{s^2+\omega^2}$
Properties		
0	$c_1 y_1(t) + c_2 y_2(t)$	$c_1 Y_1(s) + c_2 Y_2(s)$
1	$y'(t)$	$sY(s) - y(0)$
2	$ty(t)$	$-Y'(s)$
3	$e^{at}y(t)$	$Y(s-a)$
4	$u(t-a)y(t-a)$	$e^{-ac}Y(s)$
5	$y_1 * y_2(t)$	$Y_1(s)Y_2(s)$

In property 4: $u(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$. In property 5: $y_1 * y_2(t) = \int_0^t y_1(t-\tau)y_2(\tau)d\tau$