Be sure that this examination has 2 pages.

The University of British Columbia

Sessional Examinations - April 2005

Mathematics 300

Introduction to Complex Variables

Closed book examination | Time: $2\frac{1}{2}$ hours

Special Instructions: No calculators, books or notes are allowed. $\mathbf{C}$ and $\mathbf{R}$ denote the sets of complex and real numbers, respectively. $\mathbf{N} = \{1, 2, \ldots\}$.

Marks

[15] **1.** Find the Laurent series for $f(z) = \frac{z^2}{z^2-z-2}$ centered at 0 which converges when $z = \frac{3i}{2}$. Find all z for which this Laurent series converges.

[10] **2.** Carefully define:

(i) An essential singularity of a complex-valued function f defined on a domain $D \subset \mathbf{C}$.

(ii) Uniform convergence of a sequence of complex-valued functions on a domain D.

[20] **3.** Let C_r denote the circle $\{|z| = r\}$ traversed once in the counterclockwise direction. Calculate the following:

(a) $\int_\Gamma z\cos(z^2)\,dz$, where Γ is a directed contour from $i\sqrt{\frac{\pi}{2}}$ to $\sqrt{\frac{\pi}{2}}$.

(b) $\int_{C_1} \frac{1}{z(e^z-1)}\,dz$.

(c) $\int_{C_{1/2}} \mathrm{Log}(1+z)\,dz$, where Log is the principal branch of the logarithm function.

(d) $\int_{C'} \frac{e^{2z}}{(z-1)^{10}}\,dz$ where C' is the circle centered at 1 of radius 1 traversed once in the counterclockwise direction.

[10] **4.** True or False. If True, explain why. If false, provide a counter-example.

(a) If $\{c_n : n \in \mathbf{N}\}$ is a sequence of complex numbers such that $\lim_{n\to\infty} c_n = 0$ then the infinite series $\sum_{n=1}^{\infty} c_n$ is convergent.

(b) If f and g are entire functions on the complex plane and $f = g$ on $\{z \in \mathbf{C} : |z| = 1\}$ then $f = g$ on $\{z \in \mathbf{C} : |z| \leq 1\}$.

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[10] **5.** Give examples of the following (provide *brief* justifications):

(a) A power series $\sum_{n=0}^{\infty} a_n z^n$ which converges if and only if $|z| \leq 2$.

(b) A function which is analytic on $\mathbf{C}$ except for a single isolated singularity such that its Taylor series at 0 has radius of convergence 1 and its Taylor series at i has radius convergence 1.

[10] **6.** Let log denote the natural logarithm function defined on $(0, \infty)$ and taking values in $\mathbf{R}$. Let $D = \{x + iy : x > 0, y \in \mathbf{R}\}$ be the right half-plane, and for $x + iy \in D$ set $u(x, y) = \log(ax^2 + by^2)$, where a and b are positive real constants.

(a) If $a = b$ find an analytic function f on D such that $u(x, y) = Re(f(x + iy))$ for all $x + iy$ in D.

(b) If $a \neq b$ show that no such f exists.

[15] **7.** Let $f(z) = \frac{1}{1+z^3}$.

(a) If γ_R is the circular arc $\{Re^{i\theta} : 0 \leq \theta \leq \frac{2\pi i}{3}\}$ traversed in the counterclockwise direction, prove that $\lim_{R\to\infty} \int_{\gamma_R} f(z)\, dz = 0$.

(b) If Γ_R is the straight line segment from 0 to $Re^{2\pi i/3}$, prove that

$$\int_{\Gamma_R} f(z)\, dz = e^{2\pi i/3} \int_0^R f(x) dx.$$

(c) Show that $\int_0^\infty f(x)\, dx = \frac{2\pi\sqrt{3}}{9}$.

[10] **8.** (a) Let f and g be entire functions such that for some $r > 0$, $f(z) = g(z)$ for all $|z| < r$. Prove that $f(z) = g(z)$ for all $z \in \mathbf{C}$.

(b) Find all entire functions f such that $f'(z) = 1$ for $|z| = 1$.

[100] **Total Marks**

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