The University of British Columbia

Final Examination - December 2009

Mathematics 265

Section 101

Closed book examination Time: 2.5 hours

Last Name: ________________ **First:** ____________ **Signature** ________________

Student Number ________________

Special Instructions:

- Be sure that this examination has 13 pages. Write your name on top of each page.
- You are allowed to bring into the exam one $8\frac{1}{2} \times 11$ formula sheet filled on both sides. No calculators or any other aids are allowed.
- In case of an exam disruption such as a fire alarm, leave the exam papers in the room and exit quickly and quietly to a pre-designated location.

Rules governing examinations

1		10
2		10
3		10
4		15
5		20
6		20
7		15
Total		100

Some Laplace transforms:

$f(t)$	$F(s) = \mathcal{L}\{f(t)\}$
1	$\frac{1}{s}$, $s > 0$
t^n	$\frac{n!}{s^{n+1}}$, $s > 0$
e^{at}	$\frac{1}{s-a}$, $s > a$
$\sin(at)$	$\frac{a}{s^2+a^2}$, $s > 0$
$\cos(at)$	$\frac{s}{s^2+a^2}$, $s > 0$
$\sinh(at)$	$\frac{a}{s^2-a^2}$, $s > \lvert a \rvert$
$\cosh(at)$	$\frac{s}{s^2-a^2}$, $s > \lvert a \rvert$
$u_c(t)$	$\frac{e^{-cs}}{s}$, $s > 0$
$u_c(t)f(t-c)$	$e^{-cs}F(s)$
$e^{ct}f(t)$	$F(s-c)$

[10] **1**. Solve the initial value problem:

$$(e^x \sin x)y' = -1 - (e^x \cos x)y,$$

with $y(\pi/2) = e^{-\pi/2}$.

[10] **2**. Find all solutions of the differential equation

$$(x+1)^3 y' + (x+1)e^{-y} = 0.$$

[10] **3**. Consider the differential equation

$$y'' + p(t)y' + q(t)y = 0 \qquad (*)$$

where p and q are continuous functions for all t.

(a) Can $y(t) = \sin(t^2)$ be a solution on an interval containing $t = 0$ of the differential equation $(*)$? Explain your answer.

(b) Calculate the Wronskian of t and t^2.

(c) Can t and t^2 both be solutions of the same differential equation $(*)$? Explain clearly.

[15] **4**. Use the method of undetermined coefficients to find the general solution of

$$y'' + y = \cos t.$$

Extra space (if needed)

[20] **5**. Consider the initial value problem

$$y'' - 3y' + 2y = g(t), \qquad y(0) = 1,\ y'(0) = 1$$

where

$$g(t) = \begin{cases} 1 & \text{if } 0 \le t < 1 \\ 0 & \text{if } t \ge 1. \end{cases}$$

(a) Compute the Laplace transform $\mathcal{L}\{g(t)\}$.

(b) Use the method of Laplace transforms to solve the initial value problem.

Extra space (if needed)

[20] **6**. Consider the following 2×2 matrix with real coefficients

$$A = \begin{pmatrix} 3 & 1 \\ 0 & a \end{pmatrix},$$

and consider the system of differential equations

$$X'(t) = AX(t), \tag{1}$$

where $X(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$.

(a) Write the determinant $|A - \lambda I|$ in factored form and then find the eigenvalues λ_1 and λ_2 of the matrix A in terms of a.

(b) In the case where $a \neq 3$, find the eigenvectors corresponding to λ_1 and λ_2.

(c) If $a \neq 3$, find two solutions $X^{(1)}(t)$ and $X^{(2)}(t)$, in terms of a, so that $\{X^{(1)}, X^{(2)}\}$ forms a fundamental set of solutions for (1).

(d) Deduce the general solution of (1) in the case $a \neq 3$.

(e) What is/are the eigenvalue(s) of A in the case where $a = 3$? Find, in this case, fundamental solutions $X^{(1)}(t)$ and $X^{(2)}(t)$, and then the general solution of (1).

Extra space (if needed)

[15] **7**. Consider the differential equation $y'' - 4y' + 13y = 0$.

(a) Transform the above equation into a system of first order differential equations, and write it in matrix form $X'(t) = AX(t)$.

(b) Find two real-valued solutions $X^{(1)}(t)$ and $X^{(2)}(t)$ that form a fundamental set of solutions to the system $X' = AX$ from part (a). What is the general solution of the system?

(c) Describe the behaviour of the solutions as $t \to \infty$.

Extra space (if needed)