Be sure that this examination has 20 pages including this cover

The University of British Columbia
Sessional Examinations - December 2016

Mathematics 257/316
Partial Differential Equations

Closed book examination Time: $2\frac{1}{2}$ hours

Name ______________________ **Signature** ______________________

Student Number______________________ **Instructor's Name** ______________________

Section Number ______________________

Special Instructions:

One $8\frac{1}{2}$" $\times$ 11" formula sheet has been provided.
Notes are permitted only on the back of the formula sheet.
Show your work in the spaces provided.
Explain your solutions sufficiently to be understood.

Rules Governing Formal Examinations

1		20
2		20
3		20
4		20
5		20
Total		100

[20] **1.**

(a) Fourier series: Compute the Fourier sine series of $f(x)$, defined on $[0, \pi]$ by $f(x) = \cos x$. [8 marks]

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(b) Consider the following boundary value problem for Laplace's equation on half an annulus.

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0,$$

$$u(r,0) = 0, \qquad 1 < r < 2,$$

$$u(r,\pi) = 0, \qquad 1 < r < 2,$$

$$u(1,\theta) = \sin 3\theta, \qquad 0 < \theta < \pi.$$

$$u(2,\theta) = \cos\theta, \qquad 0 < \theta < \pi.$$

Find the solution using separation of variables. [12 marks]

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Question 1b continued

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[20] **2.** The motion of a string subject to a gravitational load satisfies the following initial-boundary value problem:

$$u_{tt} = a^2 u_{xx} - g, \quad 0 < x < 1, \quad t > 0 \qquad (*)$$

$$u(0,t) = u(1,t) = 0,$$

$$u(x,0) = \sin(\pi x), \quad u_t(x,0) = 0.$$

Here g is the acceleration due to gravity and a is the wave speed. Treat both as fixed constant parameters.

(a) Determine the static deflection of the string, which is determined by solving (*) subject to the boundary conditions, but setting $u_{tt} = 0$, i.e. it is a steady state. [8 marks]

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(b) Use the solution obtained in (a) to reduce the initial-boundary value problem to solving a homogeneous wave equation subject to homogeneous boundary conditions. Now use separation of variables to determine the solution to this boundary value problem and hence the complete solution of the entire initial-boundary value problem. [12 marks]

HINT: The following integral may be useful:

$$\int_0^1 (x^2 - x)\sin n\pi x \; dx = 2\frac{\cos n\pi - 1}{(n\pi)^3}$$

.

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Question 2b continued

Continued on page 8

[20] **3.** The function $f(x)$ is defined for $x \in (0,1)$ by $f(x) = x^2 - 1/3$.
The following are 4 different Fourier series that each converge to $f(x)$ on $(0,1)$

A: $\sum_{n=1}^{\infty} a_n \cos(n-1/2)\pi x, \quad a_n = \left(\dfrac{4(-1)^{n+1}}{3(n-1/2)\pi} + \dfrac{4(-1)^n}{[(n-1/2)\pi]^3}\right).$

B: $\sum_{n=1}^{\infty} b_n \sin n\pi x, \quad b_n = \left(\dfrac{4(-1)^{n+1}-2}{3n\pi} + \dfrac{4[(-1)^n-1]}{[n\pi]^3}\right).$

C: $\sum_{n=1}^{\infty} c_n \cos n\pi x, \quad c_n = \dfrac{4(-1)^n}{(n\pi)^2}.$

D: $\sum_{n=1}^{\infty} d_n \sin(n-1/2)\pi x, \quad d_n = \left(\dfrac{-2}{3(n-1/2)\pi} + \dfrac{4(-1)^{n+1}}{[(n-1/2)\pi]^2} - \dfrac{4}{[(n-1/2)\pi]^3}\right).$

(a) To what value does series B converge to at $x = 1/2$? [1 mark]

(b) To what value does series A converge to at $x = 0$? [1 mark]

(c) To what value does series D converge to at $x = 1$? [1 mark]

(d) To what value does series C converge to at $x = -1/2$? [1 mark]

HINT: you should not need to sum any series to evaluate these.

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(e) To what value does series A converge to at $x = -1/2$? [1 mark]

(f) To what value does series D converge to at $x = 1$? [1 mark]

(g) To what value does series C converge to at $x = 0$? [1 mark]

(h) To what value does series B converge to at $x = -1/2$? [1 mark]

HINT: you should not need to sum any series to evaluate these.

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Question 3 continued

(i) Use 2 of the Fourier series given to help solve the following BVP

$$u_{xx} + u_{yy} = 0, \quad 0 < x < 1,\ 0 < y < 1$$

$$u(x, 0) = x^2 - 1/3, \quad 0 < x < 1,$$

$$u(x, 1) = 0, \quad 0 < x < 1,$$

$$u_x(0, y) = 0, \quad 0 < y < 1,$$

$$u_x(1, y) = y^2 - 1/3, \quad 0 < y < 1.$$

[12 marks]

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Question 3i continued

Continued on page 12

[20] **4.** Consider the the following IBVP

$$u_t = u_{xx} - 1, \qquad 0 < x < 1,\ t \geq 0,$$

subject to the conditions: $u(0,t) = 0$, $u(1,t) = 1$, $u(x,0) = 0$.

(a) Writing $u(x,t) = u_s(x) + v(x,t)$ state the problems that are satisfied by the steady state solution, $u_s(x)$, and the transient solution $v(x,t)$. [4 marks]

(b) Find the steady state solution, $u_s(x)$. [4 marks]

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(c) Find the transient solution, $v(x,t)$ using separation of variables.
HINT: The following integral may be useful:

$$\int_0^1 (x^2 - x)\sin n\pi x\ dx = 2\frac{\cos n\pi - 1}{(n\pi)^3}.$$

[8 marks]

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Question 4c continued:

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(d) Estimate how long it takes for $|v(x,t)|$ to decay to 1% of its initial size at $x = 0.5$, i.e. when is $|v(0.5,t)| = 0.01|v(0.5,0)|$? You may base your estimate on the 1st non-zero term in the series solution for $v(x,t)$ and may leave your estimate in the form of an expression involving ln (or similar functions) as you don't have a calculator. [4 marks]

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[20] **5.** Series solutions of 2nd order differential equations.

(a) Find 2 linearly independent series solutions, $y = \sum_{k=0}^{\infty} a_k x^k$, to the DE:

$$y'' + 2xy' - y = 0.$$

What is the radius of convergence of these solutions? [7 marks]

NOTE: If you are are unable to give a general formula for the coefficients, evaluate the first 3 terms in each series and give the general recurrence relationship.

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Question 5a continued:

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(b) Using part (a) write down the general solution to

$$y'' + 2xy' - y = 0,$$

and find the solution that satisfies initial conditions $y(0) = 2$, $y'(0) = 0$. [4 marks]

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(c) Consider the DE:

$$4xy'' + 2y' + y = 0.$$

Classify the points $0 \leq x < \infty$ as either: ordinary points, regular singular points or irregular singular points. [4 marks]

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(d) Find a series solution close to $x = 0$, for the following DE

$$4xy'' + 2y' + y = 0.$$

[4 marks]

The End