The University of British Columbia

Final Examination - December 17, 2012

Mathematics 257/316

All Sections

Closed book examination Time: 2.5 hours

Last Name ______________ **First** ____________ **Signature** ______________

Student Number ______________

Special Instructions:

No books, notes, or calculators are allowed.

1		20
2		20
3		20
4		20
5		20
Total		100

1. Consider the differential equation

$$8x^2y'' + 2xy' + (1 + 2x)y = 0 \tag{1}$$

(a) Classify the points $0 \leq x < \infty$ as ordinary points, regular singular points, or irregular singular points.

(b) Find two values of r such that there are solutions of the form $y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$.

(c) Use the series expansion in (b) to determine two independent solutions of (1). You only need to calculate the first three non-zero terms in each case.

[20 marks]

(Question 1 Continued)

(Question 1 Continued)

2. Consider the following initial boundary value problem for the heat equation:

$$\begin{aligned} u_t &= u_{xx} - u\ , \qquad 0 < x < 1\ , \qquad t > 0 \\ u_x(0,t) &= 0, \qquad u_x(1,t) = 0 \\ u(x,0) &= x \end{aligned} \tag{2}$$

(a) Determine the solution to the boundary value problem (2) by separation of variables.

[14 marks]

(b) Briefly describe how you would use the method of finite differences to obtain an approximate solution this boundary value problem that is accurate to $O(\Delta x^2, \Delta t)$ terms. Use the notation $u_n^k \simeq u(x_n, t_k)$ to represent the nodal values on the finite difference mesh. Explain how you propose to approximate the boundary condition $u_x(0,t) = 1$ with $O(\Delta x^2)$ accuracy.

Hint: Taylor's expansion may prove useful: $f(x+\Delta x) = f(x) + \frac{f\prime(x)}{1!}\Delta x + \frac{f''(x)}{2!}\Delta x^2 + O(\Delta x^3)$.

[6 marks]

[total 20 marks]

(Question 2 Continued)

(Question 2 Continued)

3. The motion of a string subject to a gravitational load satisfies the following initial-boundary value problem:

$$
\begin{aligned}
u_{tt} &= c^2 u_{xx} - g, \; 0 < x < 1, \; t > 0 && (3)\\
u(0,t) &= u(1,t) \, = \, 0 && (4)\\
u(x,0) &= \sin(\pi x), \; u_t(x,0) \, = \, 0.
\end{aligned}
$$

Here g is the acceleration due to gravity, which you may assume is constant.

(a) Determine the static deflection of the string, which is determined by solving (3) in which it is assumed that $u_{tt} = 0$ subject to the boundary conditions (4). [8 marks]

(b) Use the solution obtained in (a) to reduce the initial-boundary value problem to solving a homogeneous wave equation subject to homogeneous boundary conditions. Now use separation of variables to determine the solution to this boundary value problem and hence the complete solution of the entire initial-boundary value problem.

[12 marks]

HINT: The following integral may be useful:

$$\int_0^1 \left(x^2 - x\right) \sin\left(n\pi x\right) dx = 2\frac{\cos n\pi - 1}{n^3\pi^3}$$

[total 20 marks]

(Question 3 Continued)

(Question 3 Continued)

4. (a) Consider the eigenvalue problem

$$\begin{aligned} r^2R'' + rR' + \lambda R &= 0 \\ R(1) &= 0 = R'(e^2) \end{aligned}$$

Reduce this problem to the form of a Sturm-Liouville eigenvalue problem. Determine the eigenvalues and corresponding eigenfunctions. [8 marks]

(b) Use the eigenfunctions in (a) to solve the following mixed boundary value problem for the annular region:

$$\begin{aligned} u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} &= 0, \quad 1 < r < e^2, \quad 0 < \theta < \pi \\ u(r,0) &= 0 \quad \text{and} \quad u(r,\pi) = f(r) \\ u(1,\theta) &= 0 \quad \text{and} \quad \frac{\partial}{\partial r}u(e^2,\theta) = 0 \end{aligned}$$

[12 marks]
[total 20 marks]

(Question 4 Continued)

(Question 4 Continued)

5. Solve the inhomogeneous heat conduction problem:

$$\begin{aligned} u_t &= u_{xx} + xt,\ 0 < x < 1,\ t > 0 \\ u_x(0,t) &= \frac{t^2}{2},\ \text{and } u(1,t)\ = 0 \\ u(x,0) &= 0. \end{aligned}$$

[20 marks]

(Question 5 Continued)

(Question 5 Continued)

(Additional Page)