Math 257/316 Final Exam

20 December 2010

Last Name: ____________________ First Name: ____________________

Student Number: ____________________ Signature: ____________________

Instructions. The exam lasts 2.5 hours. **No calculators or electronic devices of any kind are permitted**. A formula sheet is attached. There are **12 pages** in this test including this cover page, blank pages, and the formula sheet. Unless otherwise indicated, show all your work.

Problem #	Value	Grade
1	20	
2	20	
3	20	
4	20	
5	20	
Total	100	

1. Consider the ordinary differential equation

$$2xy'' + y' - xy = 0.$$

a) [5 marks] Classify all the points $x \geq 0$ as ordinary points, regular singular points or irregular singular points.

b) [15 marks] Find the first 3 non-zero terms of the series solution about $x = 0$ satisfying

$$y(0) = 0 \quad \text{and} \quad \lim_{x \to 0} x^{1/2} y'(x) = 1.$$

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2. Consider the wave equation

$$u_{tt} = u_{xx},$$

with initial conditions

$$u(x,0) = f(x) = \begin{cases} 2(1+x) & -1 < x < -1/2, \\ 1 & -1/2 \leq x \leq 1/2, \\ 2(1-x) & 1/2 < x < 1, \\ 0 & \text{otherwise.} \end{cases} \qquad u_t(x,0) = 0.$$

a) [10 marks] Suppose the domain is infinite, $-\infty < x < \infty$. Write down d'Alembert's solution, and carefully sketch $u(x,0)$, $u(x,1/2)$, $u(x,1)$.

b) [10 marks] Suppose instead the domain is finite, with reflecting (Neumann) boundary conditions at $x = \pm 1$:

$$u_x(-1,t) = 0, \quad u_x(1,t) = 0,$$

Briefly describe how you would use the method of finite differences to find an approximate solution to this problem. Use the notation $u_n^k \approx u(x_n, t_k)$ to denote the values of u on the finite difference mesh, and include how you propose to incorporate the boundary and initial conditions.

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3. [20 marks] Solve Laplace's equation for $u(r, \theta)$ in a quarter of an annulus:

$$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0, \quad 1 < r < 2, \ 0 < \theta < \pi/2,$$

$$u_\theta(r, 0) = 0, \quad u_\theta(r, \pi/2) = 0, \qquad u(1, \theta) = 0, \quad u(2, \theta) = \cos 4\theta.$$

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4. Consider the heat equation with a time-dependent boundary condition

$$u_t = u_{xx}, \quad 0 < x < 1, \ t > 0,$$

$$u(0,t) = 0, \quad u(1,t) = e^{-t}, \qquad u(x,0) = x.$$

a) [5 marks] By finding a suitable function $w(x,t)$ that satisfies the boundary conditions and the initial condition, show how this problem can be transformed to the following problem for $v(x,t)$:

$$v_t = v_{xx} + s(x,t), \qquad v(0,t) = 0, \quad v(1,t) = 0, \qquad v(x,0) = 0,$$

where $s(x,t)$ is a source term that you should find.

b) [15 marks] Use an eigenfunction expansion to solve for v and hence find the solution to the original problem for $u(x,t)$.

Hint: You may find it helpful to know the Fourier sine series for x:

$$x = \sum_{n=1}^{\infty} -\frac{2(-1)^n}{n\pi} \sin n\pi x \quad \textit{for} \quad 0 < x < 1.$$

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5. a) [5 marks] Find the eigenfunctions, and the equation of which the eigenvalues λ_n are roots, for the Sturm Liouville problem

$$X'' + \lambda^2 X = 0, \qquad X'(0) = 0, \quad X'(1) = -X(1).$$

b) [15 marks] Solve Laplace's equation in a semi-infinite strip:

$$u_{xx} + u_{yy} = 0, \qquad 0 < x < 1, \ 0 < y < \infty,$$

$$u_x(0, y) = 0, \quad u_x(1, y) + u(1, y) = 0, \qquad u(x, 0) = 1, \quad u(x, y) \to 0 \text{ as } y \to \infty,$$

giving your answer in terms of the eigenvalues, λ_n, and evaluating any integrals.

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