Be sure that this examination has 5 Questions and 14 pages including this cover

The University of British Columbia

Final Examinations - 7 December, 2007

Mathematics 257/316

All Sections

Time: 2.5 hours

First Name (USE CAPITALS)	____________	**Last Name** (USE CAPITALS)	____________
Signature	____________	**Instructor's Name**	____________
Student Number	____________	**Section Number**	____________

Special Instructions:

Students are not allowed to bring any notes into the exam.
No calculators are allowed.

Rules governing examinations:

1. **Each candidate should be prepared to produce his/her library/AMS card upon request.**

2. **Read and observe the following rules:**
   No candidate shall be permitted to enter the examination room after the expiration of one half hour, or to leave during the first half hour of the examination.
   Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.
   CAUTION - Candidates guilty of any of the following or similar practices shall be immediately dismissed from the examination and shall be liable to disciplinary action.
   (a) Making use of any books, papers or memoranda, other than those authorized by the examiners.
   (b) Speaking or communicating with other candidates.
   (c) Purposely exposing written papers to the view of other candidates. The plea of accident or forgetfulness shall not be received.

3. **Smoking is not permitted during examinations.**

1		25
2		20
3		15
4		20
5		20
Total		100

1. Consider the differential equation

$$3x^2y'' + 5xy' + (x-1)y = 0 \qquad (1)$$

(a) Classify the points $0 \leq x < \infty$ as ordinary points, regular singular points, or irregular singular points.

(b) Find two values of r such that there are solutions of the form $y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$.

(c) Use the series expansion in (b) to determine two independent solutions of (1). You only need to calculate the first three non-zero terms in each case.

(d) Determine the radius of convergence of the series in (c).

[25 marks]

(Question 1 Continued)

2. Find the solution to the following initial boundary value problem for the heat equation:

$$\begin{aligned} u_t &= u_{xx} + u, \quad 0 < x < 1, \quad t > 0 \\ u_x(0,t) &= 0 \text{ and } u_x(1,t) = 1 \\ u(x,0) &= \cos(2\pi x), \quad 0 < x < 1 \end{aligned}$$

Hint: $\displaystyle\int_0^1 \cos(n\pi x)\cos(x)dx = \frac{(-1)^{n+1}}{(\pi n)^2 - 1}\sin(1)$

[20 marks]

(Question 2 Continued)

3. The displacement $u(x,t)$ of a string of length 1 satisfies the wave equation $u_{tt} = c^2 u_{xx}$. The string is set in motion from its equilibrium position $u = 0$ with an initial velocity $g(x)$ while both the ends of the string are held fixed.

   (a) Write down the initial boundary value problem satisfied by the displacement $u(x,t)$ of the string, and determine the solution to this equation.

   (b) Find $u(x,t)$ when $g(x)$ is defined as follows:

$$g(x) = \begin{cases} 1 & \text{if } 0 < x < \frac{1}{2} \\ 0 & \text{if } \frac{1}{2} \leq x < 1 \end{cases}$$

[15 marks]

(Question 3 Continued)

4. Use separation of variables to solve the following boundary value problem for part of the annular region:

$$
\begin{aligned}
u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} &= 0, \quad a < r < b, \quad 0 < \theta < \pi/2 \\
u_\theta(r,0) &= 0 \text{ and } u_\theta(r,\pi/2) = 0 \\
u(a,\theta) &= 1 + \cos(4\theta) \text{ and } u(b,\theta) = 0
\end{aligned}
$$

[20 marks]

(Question 4 Continued)

5. Consider the following Sturm-Liouville boundary value problem:

$$\begin{aligned} y'' + \lambda y &= 0, \quad 0 < x < 1 \\ y'(0) &= 0 \quad \text{and} \quad y(1) + y'(1) = 0 \end{aligned} \tag{2}$$

(a) Determine the form of the eigenfunctions and the equation satisfied by the eigenvalues for boundary value problem (2).

(b) Show that there exists an infinite sequence λ_n of eigenvalues and estimate λ_n for large values of n.

(c) Now show how you would use the above eigenvalues and eigenfunctions to solve the following initial boundary value problem for the heat equation:

$$\begin{aligned} u_t &= u_{xx}, \quad 0 < x < 1, \quad t > 0 \\ u_x(0,t) &= 0 \text{ and } u(1,t) + u_x(1,t) = 0 \\ u(x,0) &= f(x), \quad 0 < x < 1 \end{aligned}$$

[20 marks]

(Question 5 Continued)

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