The University of British Columbia

Final Examination - December 9, 2013

MATH 256

Sections 102 (Ren) and 103 (Nagata)

Closed book examination Time: 2.5 hours

Name ______________________ **Signature** ______________________

Student Number ______________________ **Section Number** ____________

Special Instructions:

One 8.5"×11" sheet of notes, written or printed on both sides, is allowed.
No books or calculators or other notes are allowed.
A table of Laplace transforms is provided.

Student Conduct during Examinations

- Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCcard for identification.
- Candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.
- No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no candidate shall be permitted to enter the examination room once the examination has begun.
- Candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.
- Candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
  (a) speaking or communicating with other candidates, unless otherwise authorized;
  (b) purposely exposing written papers to the view of other candidates or imaging devices;
  (c) purposely viewing the written papers of other candidates;
  (d) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
  (e) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s)–(electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).
- Candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.
- Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.
- Candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

1		20
2		20
3		20
4		20
5		20
Total		100

[20] **1.** (a) Verify that $y_1(t) = e^t$ is a solution of the homogeneous equation

$$y'' - \frac{t}{t-1}\,y' + \frac{1}{t-1}\,y = 0, \qquad t > 1. \tag{1}$$

(b) Find another solution $y_2(t)$ of the homogeneous equation (1) which is not a constant multiple of $y_1(t)$, and verify that y_1, y_2 are linearly independent (*i.e.* form a fundamental set of solutions).

(c) Find the general solution of the nonhomogeneous equation

$$y'' - \frac{t}{t-1}\,y' + \frac{1}{t-1}\,y = (t-1)e^t, \qquad t > 1. \tag{2}$$

[20] **2.** Consider the system

$$\mathbf{x}' = \begin{bmatrix} 7 & a \\ -4 & 3 \end{bmatrix} \mathbf{x}.$$

(a) Find the general solution if $a > 1$.

(b) Find the general solution if $a = 1$.

[20] **3**. (a) Find the Laplace transform of

$$g(t) = \begin{cases} 0, & 0 \le t < 1, \\ 2, & 1 \le t < 2, \\ 0, & t \ge 2. \end{cases}$$

(b) Use the definition of the Laplace transform $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$, integration by parts, and properties of the δ-function (or unit impulse function) to show that

$$\mathcal{L}\{\delta'(t-c)\} = se^{-cs}, \quad \text{if } c > 0.$$

(c) Solve the initial value problem

$$y'' + 4y = g(t) - \tfrac{1}{2}\delta'(t-3), \qquad y(0) = \tfrac{1}{2}, \quad y'(0) = 0,$$

where $g(t)$ is as given in part (a). Your final expression for $y(t)$ should be defined piecewise on the four intervals $0 \le t < 1$, $1 \le t < 2$, $2 \le t < 3$, and $t \ge 3$.

[20] **4**. (a) Let

$$f(x) = \begin{cases} 0, & -2 < x < 0, \\ 1, & 0 \le x \le 1, \\ 0, & 1 < x \le 2. \end{cases}$$

Let f be extended elsewhere as a periodic function of period 4 (*i.e.* $f(x+4) = f(x)$ for all $-\infty < x < \infty$). Find the Fourier series of the periodic function of period 4.

(b) Let

$$f(x) = \pi - x \qquad \text{for} \qquad 0 < x < \pi.$$

Let f first be extended so it is continuous on the closed interval $[\,0, \pi\,]$, and then extended into the interval $(\pi, 2\pi\,]$ so that it is odd about $x = \pi$ (*i.e.* $f(2\pi - x) = -f(x)$ for all $0 \le x \le 2\pi$). Next let the resulting function be extended into the interval $(-2\pi, 0)$ as an even function (*i.e.* $f(-x) = f(x)$ for all $-2\pi < x < 2\pi$). Finally let the resulting function be extended elsewhere as a periodic function of period 4π (*i.e.* $f(x + 4\pi) = f(x)$ for all $-\infty < x < \infty$). Find the Fourier series of the periodic function of period 4π.

[20] **5**. (a) Find all possible eigenvalues λ_n and corresponding eigenfunctions $y_n(x)$ of

$$y'' + \lambda y = 0, \qquad y'(0) = 0, \quad y(\pi) = 0.$$

Note that at $x = 0$ the boundary condition is on the derivative y', and at $x = \pi$ the boundary condition is on y.

(b) Use separation of variables to solve the heat equation

$$u_t = u_{xx} \qquad \text{for} \quad 0 < x < \pi, \quad t > 0,$$

subject to the boundary conditions

$$u_x(0,t) = 0, \quad u(\pi,t) = 0 \qquad \text{for} \quad t > 0,$$

and initial condition

$$u(x,0) = f(x) \qquad \text{for} \quad 0 < x < \pi.$$

Note that at $x = 0$ the boundary condition is on the derivative u_x, and at $x = \pi$ the boundary condition is on u.

The End