Math 256 Section 201 Final Exam
Spring 2007
Instructor: Paul A.C. Chang

Last Name: ____

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Student Number: ____

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INSTRUCTIONS:

- Do not lift the cover page until instructed!
- Write your last name, first name, student number, and email address in the spaces above.
- No calculators allowed.
- This exam consists of 10 questions on 16 pages (including this one).
- The maximum score on this exam is 100.
- You have 180 minutes to complete this exam.
- Good Luck!

1) Answer the following questions. You need not show work for this section.

A) What is $-2+3$? (1 mark)

B) Spencer, Alana, and Jacob equally share 1242 gumballs. How many does each kid get? (1 mark)

C) True or False: The equation $dx + dy = 0$ is exact. (1 mark)

D) True or False: $x_0 = 0$ is an ordinary point of the ODE $x^2 y'' + y = 0$. (1 mark)

E) True or False: Metals have low thermal conductivity. (1 mark)

F) True or False: The Special Fundamental Matrix Ψ satisfies $\Psi(t+s) = \Psi(t)\Psi(s)$. (1 mark)

G) True or False: The function $f(x) = \sinh x$ is odd. (1 mark)

H) True or False: Fick's Law describes diffusion. (1 mark)

I) True or False: Laplace's Equation in Cartesian coordinates is $u_{xx} + u_{yy} + u_{zz} = 0$. (1 mark)

J) True or False: Fourier's Law of Heat Conduction describes the spontaneous transfer of thermal energy through matter, from regions of higher temperature to lower temperature. (1 mark)

2) Let A be an $n \times n$ matrix with eigenvalue r, and corresponding eigenvector $\vec{\xi}$ and corresponding generalized eigenvector $\vec{\eta}$. Show that $\vec{x} = te^{rt}\vec{\xi} + e^{rt}\vec{\eta}$ solves $\vec{x}' = A\vec{x}$. (10 marks)

3) Solve $y'' + 4y = 4t^2 + 5e^t$, $y(0) = 5.5$, $y'(0) = 7$. (10 marks)

4) Find the solution of $u_{tt} = u_{xx}$ subject to the boundary conditions $u(0,t) = u(1,t) = 0$ and the initial conditions $u(x,0) = -x(x-1)$, $u_t(x,0) = 0$. (10 marks)

4 Cont'd)

5) Consider the following story about Romeo and Juliet. Denote

$R(t) =$ Romeo's love/hate for Juliet at time t,

$J(t) =$ Juliet's love/hate for Romeo at time t.

Positive and negative values correspond to love and hate respectively. Their story is described by the pair of ODEs

$$R' = aR + bJ,$$

$$J' = bR + aJ,$$

for some constants $a < 0$ and $b > 0$.

5a) Give a physical interpretation of the ODEs. (4 marks)

5b) Take $a = -2$ and $b = 3$. Plot a phase portrait for the ODEs. Under what initial conditions do Romeo and Juliet both fall in love with each other? (6 marks)

6) Show that $x_0 = 0$ is a regular singular point of the ODE $x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0$. (2 marks)

6b) Solve $x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0$ near $x_0 = 0$. (8 marks)

6 Cont'd)

7) Consider the equation $au_{xx} - bu_t + cu = 0$, where a, b, c are constants. By a suitable change of variables, reduce this equation to a heat equation. (10 marks)

8) Show that $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$. (10 marks)

Hint: Consider the Fourier series of $f(x) = x$ *on* $[-1,1]$.

9) Consider the modified wave equation

$$u_{tt} + u = u_{xx}, 0 < x < 1, t > 0$$

with the boundary conditions

$$u(0, t) = 0, u(1, t) = 0, t > 0$$

and the initial conditions

$$u(x, 0) = f(x), u_t(x, 0) = g(x), 0 < x < 1.$$

Solve for $u = u(x, t)$. (10 marks)

9 Cont'd)

10) Find the steady state temperature distribution T on a disk of radius a which satisfies the boundary condition

$$T_r(a, \theta) = g(\theta) \text{ for } 0 \leq \theta < 2\pi.$$

Note that this is a Neumann problem and that its solution is determined only up to an arbitrary additive constant. State a necessary condition on $g(\theta)$ for this problem to be solvable by the method of separation of variables. What does this condition mean physically? (10 marks)

10 Cont'd)